Derive the reciprocal rule from the chain rule (Hint start with $1 = \frac{\partial z}{\partial z}$)

$$1 = \frac{\partial z}{\partial z} \xRightarrow{\text{c.r.}} \frac{\partial z}{\partial x}\frac{\partial x}{\partial z} = 1 \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{1}{\frac{\partial x}{\partial z}}}$$

Use the chain rule and the reciprocal rule to derive the following:

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial z}{\partial y}}{\frac{\partial x}{\partial y}}$$

$$\frac{\partial z}{\partial x} \overset{\text{c.r.}}{=} \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} \xRightarrow{\text{r.r.}} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{1}{\frac{\partial x}{\partial y}} \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{\frac{\partial z}{\partial y}}{\frac{\partial x}{\partial y}}}$$

Let $z = z(x, y)$. Starting with the total differential of z, determine the difference between $\frac{dz}{dx}$ and $\frac{\partial z}{\partial x}$. What if $z = z(x)$? What if $z = z(x, y, z)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

divide by dx

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cancelto{1}{\frac{dx}{dx}} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

so $\boxed{\frac{dz}{dx} - \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{dy}{dx}}$

$z = z(x)$: $dz = \frac{\partial z}{\partial x}dx \Rightarrow \boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x}}$

$z = z(u, x, y)$: $dz = \left(\frac{\partial z}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial z}{\partial y}\right)dy \Rightarrow \frac{dz}{dx} = \left(\frac{\partial z}{\partial u}\right)\frac{du}{dx} + \left(\frac{\partial z}{\partial x}\right)\cancelto{1}{\frac{dx}{dx}} + \left(\frac{\partial z}{\partial y}\right)\frac{dy}{dx}$

$$\Rightarrow \boxed{\frac{dz}{dx} - \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}}$$