Normalization

Normalize the following functions over the given space

① $\psi(x) = 1 - x^2$ space $-1 \le x \le 1$

$\psi(x) = N\,\psi_{norm}(x)$ $\int_{-1}^{1} |\psi(x)|^2 dx = N^2 \cancel{\int_{-1}^{1} |\psi_{norm}|^2 dx}^{\,1}$

$N = \sqrt{\int_{-1}^{1} |\psi(x)|^2 dx} = \sqrt{\int_{-1}^{1} (1-x^2)^2 dx} = \sqrt{\int_{-1}^{1} (1 - 2x^2 + x^4)\, dx}$

$N = \sqrt{x - \frac{2x^3}{3} + \frac{x^5}{5} \Big|_{-1}^{1}} = \sqrt{\frac{15}{15} - \frac{10}{15} + \frac{3}{15} - \left(\frac{-15}{15} + \frac{10}{15} - \frac{3}{15}\right)} = \sqrt{\frac{16}{15}}$

so $\boxed{\psi_{norm} = \sqrt{\frac{15}{16}}\,(1-x^2)}$

② $\psi(\theta) = e^{i\alpha\theta}$ space $0 \le \theta < 2\pi$

$N = \sqrt{\int_0^{2\pi} |e^{i\alpha\theta}|^2 d\theta} = \sqrt{\int_0^{2\pi} e^{i\alpha\theta} e^{-i\alpha\theta} d\theta}$

$N = \sqrt{\int_0^{2\pi} 1\, d\theta} = \sqrt{2\pi}$ so $\boxed{\psi_{norm} = \frac{1}{\sqrt{2\pi}} e^{i\alpha\theta}}$

③ $\psi(x) = x e^{-\alpha x}$ space $0 \le x < \infty$ and $\alpha > 0$

Note: $\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$ $a > 0$, n positive integer

$N = \sqrt{\int_0^{\infty} |x e^{-\alpha x}|^2 dx} = \sqrt{\int_0^{\infty} x^2 e^{-2\alpha x} dx}$

$N = \sqrt{\frac{2!}{(2\alpha)^3}} = \sqrt{\frac{1}{4\alpha^3}}$ so $\boxed{\psi_{norm} = 2\alpha^{3/2} x e^{-\alpha x}}$