

Normalization

Normalize the following functions over the given space

① $\psi(x) = 1 - x^2$ space $-1 \leq x \leq 1$

$$\psi(x) = N \psi_{\text{norm}}(x) \quad \int_{-1}^1 |\psi(x)|^2 dx = N^2 \int_{-1}^1 |\psi_{\text{norm}}|^2 dx$$

$$N = \sqrt{\int_{-1}^1 |\psi(x)|^2 dx} = \sqrt{\int_{-1}^1 (1-x^2)^2 dx} = \sqrt{\int_{-1}^1 (1 - 2x^2 + x^4) dx}$$

$$N = \sqrt{x - \frac{2x^3}{3} + \frac{x^5}{5}} \Big|_{-1}^1 = \sqrt{\frac{15}{15} - \frac{10}{15} + \frac{3}{15} - \left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}\right)} = \sqrt{\frac{16}{15}}$$

② $\psi(\theta) = e^{i\alpha\theta}$ space $0 \leq \theta < 2\pi$

so

$$\psi_{\text{norm}} = \sqrt{\frac{15}{16}} (1-x^2)$$

$$N = \sqrt{\int_0^{2\pi} |e^{i\alpha\theta}|^2 d\theta} = \sqrt{\int_0^{2\pi} e^{i\alpha\theta} e^{-i\alpha\theta} d\theta}$$

$$N = \sqrt{\int_0^{2\pi} 1 d\theta} = \sqrt{2\pi} \quad \text{so } \psi_{\text{norm}} = \frac{1}{\sqrt{2\pi}} e^{i\alpha\theta}$$

③ $\psi(x) = x e^{-\alpha x}$ space $0 \leq x < \infty$ and $\alpha > 0$

Note: $\int_0^{\infty} x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$ $\alpha > 0$, n positive integer

$$N = \sqrt{\int_0^{\infty} |x e^{-\alpha x}|^2 dx} = \sqrt{\int_0^{\infty} x^2 e^{-2\alpha x} dx}$$

$$N = \sqrt{\frac{2!}{(2\alpha)^3}} = \sqrt{\frac{1}{4\alpha^3}} \quad \text{so } \psi_{\text{norm}} = 2\alpha^{3/2} x e^{-\alpha x}$$

