Final Exam

Lenarz - Math 223 December 14, 2005 Name: ____________________

Directions: Answer the following questions in the space provided. You may use a calculator and a 4"x6" note card, handwritten on both sides. If you do not show your work, you will receive no credit. The point value of each question is indicated. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!**

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1. (a) (5 points) Find parametric equations for the line through $(1, 0, -1)$ parallel to the vector from $(2, 1, 0)$ to $(1, -1, 1)$.

   The direction of the line is given by

   $$\mathbf{v} = \langle 2 - 1, -1 - 1, 0 - 1 \rangle = \langle 1, 2, -1 \rangle$$

   So the parametric equations of the line are

   $$\begin{aligned} x &= 1 + t \\ y &= 2t \\ z &= -1 - t \end{aligned}$$

   (b) (5 points) Find an equation for the plane through the $(-1, 2, 1)$ that is perpendicular to the line of part (a).

   The normal vector of the plane is given by the direction vector of the line in part (a), so

   $$\mathbf{n} = \langle 1, 2, -1 \rangle$$

   Thus the equation of the plane is

   $$\begin{aligned} (1)(x - (-1)) + 2(y - 2) + (-1)(z - 1) &= 0 \\ x + 1 + 2y - 4 - z + 1 &= 0 \\ x + 2y - z &= 2 \end{aligned}$$

2. (a) (5 points) Compute the gradient of $f(x, y, z) = xy^2 + e^x \cos y - z$ at $(0, 0, 1)$.

   $$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_(x, y, z), f_z(x, y, z) \rangle \\ &= \langle y^2 + e^x \cos y, 2xy - e^x \sin y, -1 \rangle \end{aligned}$$

   So we have

   $$\begin{aligned} \nabla f(0, 0, 1) &= \langle (0)^2 + e^0 \cos(0), 2(0)(0) - e^0 \sin(0), -1 \rangle \\ &= \langle 1, 0, -1 \rangle \end{aligned}$$

(b) (5 points) Find the equation of the plane tangent to the surface $z = xy^2 + e^x \cos y$ at $(0, 0, 1)$.

Let

$$f(x, y, z) = xy^2 + e^x \cos y - z$$

so our surface is $f(x, y, z) = 0$ and thus the normal vector of the tangent plane is given by the gradient. So from part(a) we have

$$\mathbf{n} = \nabla f(0, 0, 1) = \langle 1, 0, -1 \rangle$$

and the equation of the tangent plane is

$$\begin{aligned} (1)(x-0) + (0)(y-0) + (-1)(z-1) &= 0 \\ x - z + 1 &= 0 \\ x - z &= -1 \end{aligned}$$

3. (15 points) Evaluate $\oint_C (x^2 + 4xy)\ dx + (2x^2 + 3y)\ dy$, where C is the ellipse with counterclockwise orientation given by the parametrization $x = 4\cos t$, $y = 3\sin t$ where $0 \le t \le 2\pi$.

Since the curve is a simple, closed curve with counterclockwise orientation, we can use Green's Theorem. Notice

$$P(x, y) = x^2 + 4xy \qquad Q(x, y) = 2x^2 + 3y$$

so by Green's Theorem

$$\begin{aligned} \oint_C (x^2 + 4xy)\ dx + (2x^2 + 3y)\ dy &= \int\int_S Q_x(x, y) - P_y(x, y)\ dA \\ &= \int\int_S (4x - 4x)\ dA \\ &= \int\int_S 0\ dA \\ &= 0 \end{aligned}$$

4. (10 points) Find a vector in the direction in which

$$f(x,y,z) = \frac{360}{\sqrt{x^2+y^2+z^2}}$$

increases most rapidly at $(1,2,2)$. What is the rate of change in this direction?

The direction of greatest increase is given by the gradient at the point and the maximum rate of change is the magnitude of the gradient at the point. So we compute

$$\begin{aligned}
\nabla f(x,y,z) &= \left\langle \frac{-360x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-360y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-360z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle \\
&= -\frac{360}{(x^2+y^2+z^2)^{\frac{3}{2}}}\langle x,y,z\rangle \\
\nabla f(1,2,2) &= -\frac{360}{(1^2+2^2+2^2)^{\frac{3}{2}}}\langle 1,2,2\rangle \\
&= -\frac{360}{27}\langle 1,2,2\rangle \\
&= -\frac{40}{3}\langle 1,2,2\rangle \\
||\nabla f(1,2,2)|| &= \left|-\frac{40}{3}\right|\sqrt{1^2+2^2+2^2} \\
&= \frac{40}{3}(3) \\
&= 40
\end{aligned}$$

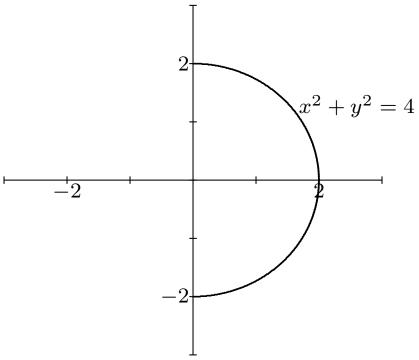
So the maximum rate of change occurs in the direction of $\nabla f(1,2,2) = -\frac{40}{3}\langle 1,2,2\rangle$ with rate $||\nabla f(1,2,2)|| = 40$.

5. (10 points) Set up (but do not evaluate) an integral in spherical coordinates for the mass of the solid lying inside the sphere $x^2+y^2+z^2 = 16$ and outside the cone $z = \sqrt{x^2+y^2}$ if the density function is $\rho(x,y,z) = xy$.

$$\begin{aligned}
m &= \int\int\int_E \rho(x,y,z)\ dV \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^4 \rho^4\sin^3\phi\cos\theta\sin\theta\ d\rho\ d\theta\ d\phi
\end{aligned}$$

6. (10 points) Reverse the order of integration in the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\;dx\;dy$$



$$\int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\;dy\;dx$$

7. For the vector field $\mathbf{F}(x,y,z)=\langle yz+1, xz+1, xy+1\rangle$

   (a) (15 points) Either find a potential function f so that $\mathbf{F}=\nabla f$ or explain why no such potential function exists.

$$\begin{aligned}
\text{curl }\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+1 & xz+1 & xy+1 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz+1 & xy+1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz+1 & xy+1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz+1 & xz+1 \end{vmatrix}\mathbf{k} \\
&= [\frac{\partial}{\partial y}(xy+1) - \frac{\partial}{\partial z}(xz+1)]\mathbf{i} - [\frac{\partial}{\partial x}(xy+1) - \frac{\partial}{\partial z}(yz+1)]\mathbf{j} \\
&\qquad + [\frac{\partial}{\partial x}(xz+1) - \frac{\partial}{\partial y}(yz+1)]\mathbf{k} \\
&= [x-x]\mathbf{i} - [y-y]\mathbf{j} + [z-z]\mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

So $\mathbf{F}(x,y,z)$ is a conservative vector field. Now we find the potential function. We want to find $f(x,y,z)$ so that $\mathbf{F}(x,y,z)=\nabla f(x,y,z)$ which means that

$$\begin{aligned}
f_x(x,y,z) &= yz+1 \\
f_y(x,y,z) &= xz+1 \\
f_z(x,y,z) &= xy+1
\end{aligned}$$

So we have

$$\begin{aligned} f(x,y,z) &= \int f_x(x,y,z)\ dx \\ &= \int yz+1\ dx \\ &= xyz+x+g(y,z) \end{aligned}$$

This means that $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz+1$ so we have

$$\begin{aligned} xz+g_y(y,z) &= xz+1 \\ g_y(y,z) &= 1 \\ \int g_y(y,z)\ dy &= \int 1\ dy \\ g(y,z) &= y+h(z) \end{aligned}$$

So $f(x,y,z) = xyz+x+y+h(z)$. This means that $f_z(x,y,z) = xy+h'(z)$. But $f_z(x,y,z) = xy+1$, so we have

$$\begin{aligned} xy+h'(z) &= xy+1 \\ h'(z) &= 1 \\ \int h'(z)\ dz &= \int 1\ dz \\ h(z) &= z+C \end{aligned}$$

So

$$f(x,y,z) = xyz+x+y+z+C$$

(b) (5 points) Calculate the work done by $\mathbf{F}(x,y,z)$ in moving a particle along the path consisting of line segments from $(0,0,0)$ to $(2,0,0)$ to $(2,3,0)$ to $(2,3,4)$.

By the Fundamental Theorem for Line Integrals, we have

$$\begin{aligned} W &= \int \mathbf{F}\cdot\ d\mathbf{r} \\ &= f(2,3,4)-f(0,0,0) \\ &= [(2)(3)(4)+(2)+(3)+(4)]-[(0)(0)(0)+(0)+(0)+(0)] \\ &= 33 \end{aligned}$$

8. (15 points) Find and classify the critical points of $f(x, y) = 2xy - \frac{1}{2}x^4 - \frac{1}{2}y^4 + 1$.

We find the critical numbers by finding places where $\nabla f = \mathbf{0}$ or where ∇f does not exist.

$$\nabla f(x, y) = \langle 2y - 2x^3, 2x - 2y^3 \rangle$$

So we have the system of equations

$$\begin{aligned} 2(y - x^3) &= 0 \\ 2(x - y^3) &= 0 \end{aligned}$$

Using the first equation, we see that $y = x^3$. So the second equation becomes

$$\begin{aligned} 2(x - (x^3)^3) &= 0 \\ 2(x - x^9) &= 0 \\ 2x(1 - x^8) &= 0 \\ 2x(1 - x^4)(1 + x^4) &= 0 \\ 2x(1 - x^2)(1 + x^2)(1 + x^4) &= 0 \\ 2x(1 - x)(1 + x)(1 + x^2)(1 + x^4) &= 0 \end{aligned}$$

So $x = 0, 1, -1$. If $x = 0$, $y = 0^3 = 0$ so $(0, 0)$ is a critical point. If $x = 1$, $y = 1^3 = 1$ so $(1, 1)$ is a critical point. If $x = -1$, $y = (-1)^3 = -1$ so $(-1, -1)$ is a critical point. Now we use the second partials test.

$$\begin{aligned} f_{xx}(x, y) &= -6x^3 \\ f_{yy}(x, y) &= -6y^3 \\ f_{xy}(y, x) &= 2 \\ D(x, y) &= f^2_{xx}(x, y) f^2_{yy}(x, y) - f^2_{xy}(x, y) \\ &= (-6x^2)(-6y^2) - 2^2 \\ &= 36x^2y^2 - 4 \end{aligned}$$

So we have

Point	$D(x, y)$	$f_{xx}(x, y)$	Conclusion
$(0, 0)$	-4		Saddle Point
$(1, 1)$	32	-6	Local Max
$(-1, -1)$	32	-6	Local Max