Exam #2

Lenarz - Math 223 November 2, 2005 Name: ____________________

Directions: Answer the following questions in the space provided. You may use a calculator and a 3"x5" note card, handwritten on both sides. If you do not show your work, you will receive no credit. The point value of each question is indicated. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!**

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1. Compute the first partial derivatives of the following
   (a) (5 points) $f(x, y, z) = -5x^3y^2 + z^3 - x$

$$\begin{aligned} f_x(x,y,z) &= -15x^2y^2 - 1 \\ f_y(x,y,z) &= -10x^3y \\ f_z(x,y,z) &= 3z^2 \end{aligned}$$

   (b) (5 points) $g(x, y) = ye^{x^2+y^3}$

$$\begin{aligned} g_x(x,y) &= y(2xe^{x^2+y^3}) \\ &= 2xye^{x^2+y^3} \\ g_y(x,y) &= y(3y^2e^{x^2+y^3}) + (1)e^{x^2+y^3} \\ &= (3y^3+1)e^{x^2+y^3} \end{aligned}$$

2. (10 points) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$$

does not exist. Along the x-axis (i.e. the path $y = 0$) we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} &= \lim_{(x,0)\to(0,0)} \frac{x^2+(0)^2}{x^2-(0)^2} \\ &= \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} \\ &= \lim_{(x,0)\to(0,0)} 1 \\ &= 1 \end{aligned}$$

However, along the y-axis (i.e. the path $x = 0$) we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} &= \lim_{(0,y)\to(0,0)} \frac{(0)^2+y^2}{(0)^2-y^2} \\ &= \lim_{(0,y)\to(0,0)} \frac{y^2}{-y^2} \\ &= \lim_{(0,y)\to(0,0)} -1 \\ &= -1 \end{aligned}$$

Since the limit has different values along different paths, the limit does not exist.

3. Consider the function $f(x,y) = x^2y + xy^2$

(a) (10 points) Calculate $\nabla f(x,y)$

$$\begin{aligned} f_x(x,y) &= 2xy + y^2 \\ f_y(x,y) &= x^2 + 2xy \\ \nabla f(x,y) &= \langle f_x(x,y), f_y(x,y) \rangle \\ &= \langle 2xy + y^2, x^2 + 2xy \rangle \end{aligned}$$

(b) (10 points) Find the directional derivative of $f(x,y)$ at the point $(1,-1)$ in the direction of $\mathbf{a} = \langle 3, -4 \rangle$. First need a unit vector in the direction of $\mathbf{a}$:

$$\begin{aligned} \mathbf{u} &= \frac{1}{||\mathbf{a}||}\mathbf{a} \\ &= \frac{1}{\sqrt{3^2 + (-4)^2}}\langle 3, -4 \rangle \\ &= \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \end{aligned}$$

Then we evaluate $\nabla f(x,y)$ at our point:

$$\begin{aligned} \nabla f(1,-1) &= \langle 2(1)(-1) + (-1)^2, (1)^2 + 2(1)(-1) \rangle \\ &= \langle -1, -1 \rangle \end{aligned}$$

So we have

$$\begin{aligned} D_{\mathbf{u}}f(1,-1) &= \mathbf{u} \cdot \nabla f(1,-1) \\ &= \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \cdot \langle -1, -1 \rangle \\ &= \left(\frac{3}{5}\right)(-1) + \left(-\frac{4}{5}\right)(-1) \\ &= -\frac{3}{5} + \frac{4}{5} \\ &= \frac{1}{5} \end{aligned}$$

(c) (5 points) What is the maximum rate of change of $f(x,y)$ at the point $(1,-1)$? In what direction does it occur?

The maximum rate of change of $f(x,y)$ at $(1,-1)$ is given by

$$\begin{aligned} ||\nabla f(1,-1)|| &= ||\langle -1, -1 \rangle|| \\ &= \sqrt{(-1)^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

It occurs in the direction of $\nabla f(1,-1) = \langle -1, -1 \rangle$.

4. Suppose $\mathbf{u} = \langle 1, 0\rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}f(a,b) = 3$ and $D_{\mathbf{v}}f(a,b) = \sqrt{2}$

   (a) (10 points) Find $\nabla f(a,b)$.

$$\begin{aligned}
3 &= D_{\mathbf{u}}f(a,b) \\
&= \nabla f(a,b) \cdot \mathbf{u} \\
&= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle 1, 0\rangle \\
&= (f_x(a,b))(1) + (f_y(a,b))(0) \\
&= f_x(a,b) \\
\sqrt{2} &= D_{\mathbf{v}}f(a,b) \\
&= \nabla f(a,b) \cdot \mathbf{v} \\
&= \langle f_x(a,b), f_y(a,b)\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\
&= (f_x(a,b))\left(\frac{1}{\sqrt{2}}\right) + (f_y(a,b))\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{3}{\sqrt{2}} + \frac{f_y(a,b)}{\sqrt{2}} \\
1 &= 3 + f_y(a,b) \\
-1 &= f_y(a,b)
\end{aligned}$$

   So we have

$$\nabla f(a,b) = \langle 3, -1\rangle$$

   (b) (5 points) What is the maximum possible value of $D_{\mathbf{w}}f(a,b)$ for any vector $\mathbf{w}$?
   The maximum value is the magnitude of the gradient vector at (a,b)

$$\begin{aligned}
||\nabla f(a,b)|| &= ||\langle 3, -1\rangle|| \\
&= \sqrt{(3)^2 + (-1)^2} \\
&= \sqrt{9+1} \\
&= \sqrt{10}
\end{aligned}$$

   (c) (10 points) Find a unit vector $\mathbf{w} = \langle w_1, w_2\rangle$ such that $D_{\mathbf{w}}f(a,b) = 0$
   We must have

$$\begin{aligned}
0 &= D_{\mathbf{w}}f(a,b) \\
&= \nabla f(a,b) \cdot \mathbf{w} \\
&= \langle 3, -1\rangle \cdot \langle w_1, w_2\rangle \\
&= 3w_1 - w_2
\end{aligned}$$

   and $\mathbf{w}$ must be a unit vector, so we solve the system

$$\begin{aligned}
3w_1 - w_2 &= 0 \\
w_1^2 + w_2^2 &= 1
\end{aligned}$$

The first equation says that $3w_1 = w_2$, so we substitute that into the second equation

$$\begin{aligned} w_1^2 + (3w_1)^2 &= 1 \\ w_1^2 + 9w_2^2 &= 1 \\ 10w_1^2 &= 1 \\ w_1^2 &= \frac{1}{10} \\ w_1 &= \pm\frac{1}{\sqrt{10}} \end{aligned}$$

and $w_2 = 3\left(\pm\frac{1}{\sqrt{10}}\right) = \pm\frac{3}{\sqrt{10}}$, so either

$$\mathbf{w} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \qquad \text{or} \qquad \mathbf{w} = \left\langle -\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle$$

5. (10 points) Use the chain rule to find $\dfrac{\partial w}{\partial t}$ if $w = xy^2 - yz^2 + x^2z$, $x = s + t$, $y = s - t$ and $z = s^2$.

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\
&= (y^2 + 2xz)(1) + (2xy - z^2)(-1) \\
&= (s-t)^2 + 2(s+t)(s^2) - 2(s+t)(s-t) + (s^2)^2 \\
&= s^2 - 2st + t^2 + 2s^3 + 2s^2t - 2s^2 + 2t^2 + s^4 \\
&= s^4 + 2s^3 - s^2 + 2s^2t - 2st + 3t^2
\end{aligned}$$

6. Consider the sphere $x^2 + y^2 + z^2 = 9$.

   (a) (5 points) Find the plane tangent to the sphere at the point $(3, 0, 0)$.
   Let $F(x, y, z) = x^2 + y^2 + z^2 - 9$. Then

$$\nabla F(x, y, z) = \langle 2x, 2y, 2z \rangle$$

   The normal vector for the plane tangent to the surface at $(3, 0, 0)$ is given by

$$\begin{aligned}
\mathbf{n} &= \nabla F(3, 0, 0) \\
&= \langle 6, 0, 0 \rangle
\end{aligned}$$

   So the equation of the plane is

$$\begin{aligned}
6(x-3) + (0)(y-0) + (0)(z-0) &= 0 \\
6x - 18 &= 0 \\
6x &= 18 \\
x &= 3
\end{aligned}$$

   (b) (5 points) Find the plane tangent to the surface at the point $(2, 2, 1)$ The normal vector for the plane tangent to the surface at $(2, 2, 1)$ is given by

$$\begin{aligned}
\mathbf{n} &= \nabla F(2, 2, 1) \\
&= \langle 4, 4, 2 \rangle
\end{aligned}$$

   So the equation of the plane is

$$\begin{aligned}
4(x-2) + 4(y-2) + 2(z-1) &= 0 \\
4x - 8 + 4y - 8 + 2z - 2 &= 0 \\
4x + 4y + 2z &= 18 \\
2x + 2y + z &= 9
\end{aligned}$$

7. (10 points) Find and classify all critical points of $f(x,y) = x^2 + y^2 + x^2y + 4$. Notice there are no boundary points to consider. To find our stationary points, we look at

$$\nabla f(x,y) = \langle 2x + 2xy, 2y + x^2 \rangle$$

Stationary points occur when $\nabla f(x,y) = \mathbf{0}$, so we solve the system of equations:

$$\begin{aligned} 2x + 2xy &= 0 \\ 2y + x^2 &= 0 \end{aligned}$$

We can solve the second equation for y to get $y = -\dfrac{x^2}{2}$. We then substitute this into the first equation:

$$\begin{aligned} 2x + 2x\left(-\frac{x^2}{2}\right) &= 0 \\ 2x - x^3 &= 0 \\ -x(x^2 - 2) &= 0 \end{aligned}$$

So we have $x = 0$, $x = \sqrt{2}$ and $x = -\sqrt{2}$. If $x = 0$, then $y = -\dfrac{(0)^2}{2} = 0$, giving us the stationary point $(0,0)$. If $x = \sqrt{2}$, then $y = -\dfrac{(\sqrt{2})^2}{2} = -1$, giving us the stationary point $(\sqrt{2}, -1)$. If $x = -\sqrt{2}$, then $y = -\dfrac{(-\sqrt{2})^2}{2} = -1$, giving us the stationary point $(-\sqrt{2}, -1)$. (Notice there are no singular points since ∇f is defined everywhere.)
Now we calculate $D(x,y)$ and apply the second partials test:

$$\begin{aligned} D(x,y) &= f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y) \\ &= (2+2y)(2) - (2x)^2 \\ &= 4 + 4y - 4x^2 \end{aligned}$$

Critical Point	$D(x,y)$	$f_{xx}(x,y)$	Conclusion
$(0,0)$	4	2	local min
$(\sqrt{2},-1)$	-8	0	saddle point
$(-\sqrt{2},-1)$	-8	0	saddle point

8. (**10 BONUS POINTS**) Use Lagrange's method to find the point on the surface

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \qquad x, y, z > 0$$

that is closest to the origin.

To minimize distance, it suffices to minimize the square of the distance. The square of the distance between a point (x, y, z) and the origin is given by

$$f(x, y, z) = x^2 + y^2 + z^2$$

Our constraint is

$$g(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

First we find

$$\begin{aligned} \nabla f(x, y, z) &= \langle 2x, 2y, 2z \rangle \\ \nabla g(x, y, z) &= \left\langle -\frac{1}{x^2}, -\frac{1}{y^2}, -\frac{1}{z^2} \right\rangle \end{aligned}$$

So we set up the system

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned}$$

Which gives us

$$\begin{aligned} 2x &= -\frac{\lambda}{x^2} \\ 2y &= -\frac{\lambda}{y^2} \\ 2z &= -\frac{\lambda}{z^2} \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 1 \end{aligned}$$

Since we know $x, y, z > 0$, we can simplify the system to

$$\begin{aligned} 2x^3 &= -\lambda \\ 2y^3 &= -\lambda \\ 2z^3 &= -\lambda \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 1 \end{aligned}$$

The first three equations say that $x^3 = y^3 = z^3$, which is equivalent to $x = y = z$. Substituting this into the fourth equation gives

$$\begin{aligned} \frac{1}{x} + \frac{1}{x} + \frac{1}{x} &= 1 \\ \frac{3}{x} &= 1 \\ 3 &= x \end{aligned}$$

So our only critical point is $(3, 3, 3)$. We must verify that it is in fact a minimum, so we need to find another point that satisfies the constraint and compare the values of f. The point $(2, 3, 6)$ satisfies $g(x, y, z) = 0$, and we have

$$\begin{aligned} f(3,3,3) &= 3^2 + 3^2 + 3^2 \\ &= 27 \\ f(2,3,6) &= 2^2 + 3^2 + 6^2 \\ &= 49 \end{aligned}$$

So $(3, 3, 3)$ is in fact a global minimum.