Lenarz | Exam #1 | Name: ____________
Math 223 | October 7, 2005 | Section: ____________

Directions: Answer the following questions in the space provided. You may use a calculator and one 3″ × 5″ index card (handwritten, both sides). If you do not show your work, you will receive no credit. The point value of each question is indicated. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!**

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1. (14 points) Find the angle (to the nearest degree) between the planes

$$x - 3y + 6z = 4 \qquad\qquad -5x - y + z = 4$$

The angle between the normal vectors is the angle between the planes:

$$\mathbf{n}_1 = \langle 1, -3, 6\rangle \qquad\qquad \mathbf{n}_2\langle -5, -1, 1\rangle$$

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1||||\mathbf{n}_2||} \\
&= \frac{(1)(-5) + (-3)(-1) + (6)(1)}{\sqrt{(1)^2 + (-3)^2 + (6)^2}\sqrt{(-5)^2 + (-1)^2 + (1)^2}} \\
&= \frac{-5 + 3 + 6}{\sqrt{1 + 9 + 36}\sqrt{25 + 1 + 1}} \\
&= \frac{4}{\sqrt{46}\sqrt{27}} \\
&= \frac{4}{3\sqrt{138}} \\
\theta &= \cos^{-1}\left(\frac{4}{3\sqrt{138}}\right) \\
&\approx 83^\circ
\end{aligned}$$

2. (14 points) Find the equation of the plane containing the points $P(1, 2, 3)$, $Q(2, 3, 1)$, and $R(0, -2, -1)$.
Find two vectors in the plane

$$\begin{aligned}
\mathbf{a} &= \overrightarrow{PQ} \\
&= \langle 2 - 1, 3 - 2, 1 - 3\rangle \\
&= \langle 1, 1, -2\rangle \\
\mathbf{b} &= \overrightarrow{PR} \\
&= \langle 0 - 1, -2 - 2, -1 - 3\rangle \\
&= \langle -1, -4, -4\rangle
\end{aligned}$$

The cross product of these two vectors will be normal to the plane

$$\begin{aligned}
\mathbf{n} &= \mathbf{a}\times\mathbf{b} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & -4 & -4 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -2 \\ -4 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & -4 \end{vmatrix}\mathbf{k} \\
&= [(1)(-4)-(-4)(-2)]\mathbf{i} - [(1)(-4)-(-1)(-2)]\mathbf{j} + [(1)(-4)-(-1)(1)]\mathbf{k} \\
&= [-4-8]\mathbf{i} - [-4-2]\mathbf{j} + [-4-(-1)]\mathbf{k} \\
&= -12\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}
\end{aligned}$$

So the equation of the plane is

$$\begin{aligned}
(-12)(x-1)+(6)(y-2)+(-3)(z-3) &= 0 \\
-12x+12+6y-12-3z+9 &= 0 \\
-12x+6y-3z &= -9 \\
4x-2y+z &= 3
\end{aligned}$$

3. (10 points) Let **a**, **b**, **c**, and **d** be vectors, with c a scalar. State if the expression is a scalar quantity, a vector quantity, or meaningless.

(a) $||\mathbf{a}||$ - scalar

(b) $\mathbf{a}+c$ - meaningless

(c) $\mathbf{a}\cdot\mathbf{b}$ - scalar

(d) $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ - meaningless

(e) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ - vector

(f) $\mathbf{a}\times\mathbf{c}$ - vector

(g) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ - scalar

(h) $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ - meaningless

(i) $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ - meaningless

(j) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ - scalar

4. (14 points) Find parametric equations for the line tangent to

$$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k}$$

at the point $(2, 0, 0)$.
First notice that at the point $(2, 0, 0)$ that

$$2\cos t = 2 \qquad 2\sin t = 0 \qquad t = 0$$

so at this point $t = 0$. The direction vector of the tangent line is the derivative of the function evaluated at this point

$$\begin{aligned} \mathbf{r}'(t) &= (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k} \\ \mathbf{v} &= \mathbf{r}'(0) \\ &= -2\sin(0)\mathbf{i} + 2\cos(0)\mathbf{j} + \mathbf{k} \\ &= 2\mathbf{j} + \mathbf{k} \end{aligned}$$

So the parametric equations of the line are

$$x = 2 \qquad y = 2t \qquad z = t$$

5. Find the velocity and position function of a particle with acceleration $\mathbf{a}(t) = 4\mathbf{i}$, initial velocity $\mathbf{v}(0) = 4\mathbf{j}$ and initial position $\mathbf{r}(0) = 2\mathbf{j}$.

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{a}(t)\ dt \\ &= \int 4\mathbf{i}\ dt \\ &= 4t\mathbf{i} + \mathbf{C} \end{aligned}$$

Since $4\mathbf{j}$ is the initial velocity we have

$$\begin{aligned} 4\mathbf{j} &= \mathbf{v}(0) \\ &= 4(0)\mathbf{i} + \mathbf{C} \\ &= \mathbf{C} \end{aligned}$$

So the constant vector $\mathbf{C} = 4\mathbf{j}$ and we have

$$\mathbf{v}(t) = 4t\mathbf{i} + 4\mathbf{j}$$

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t)\ dt \\ &= \int 4t\mathbf{i} + 4\mathbf{j}\ dt \\ &= 2t^2\mathbf{i} + 4t\mathbf{j} + \mathbf{C} \end{aligned}$$

Since $2\mathbf{j}$ is the initial position we have

$$\begin{aligned} 2\mathbf{j} &= \mathbf{r}(0) \\ &= 2(0)^2\mathbf{i} + 4(0)\mathbf{j} + \mathbf{C} \\ &= \mathbf{C} \end{aligned}$$

So the constant vector $\mathbf{C} = 2\mathbf{j}$ and we have

$$\mathbf{r}(t) = 2t^2\mathbf{i} + (4t+2)\mathbf{j}$$

6. Convert $x^2 + y^2 = 16$ from rectangular coordinates to
   (a) (5 points) cylindrical coordinates
   Since $r^2 = x^2 + y^2$ we have
   $$r^2 = 16$$
   or
   $$r = 4$$
   (b) (5 points) spherical coordinates
   Since $x = \rho \sin\phi \cos\theta$ and $y = \rho \sin\phi \sin\theta$, we have
   $$\begin{aligned} x^2 + y^2 &= 16 \\ (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 &= 16 \\ \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta &= 16 \\ \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) &= 16 \\ \rho^2\sin^2\phi(1) &= 16 \\ \rho^2\sin^2\phi &= 16 \\ \rho\sin\phi &= 4 \\ \rho &= 4\csc\phi \end{aligned}$$

7. Let the position of a particle be given by
   $$\mathbf{r}(t) = \langle \sin 2t, 3t, \cos 2t\rangle \qquad -\pi \le t \le \pi$$
   (a) (6 points) Show that the velocity and acceleration of the particle are always orthogonal.
   Note that
   $$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) \\ &= \langle 2\cos 2t, 3, -2\sin 2t\rangle \\ \mathbf{a}(t) &= \mathbf{r}''(t) \\ &= \langle -4\sin 2t, 0, -4\cos 2t\rangle \\ \mathbf{v}(t)\cdot\mathbf{a}(t) &= (2\cos 2t)(-4\sin 2t) + (3)(0) + (-2\sin 2t)(-4\cos 2t) \\ &= -8\sin 2t\cos 2t + 0 + 8\sin 2t\cos 2t \\ &= 0 \end{aligned}$$

Since $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$, the two vectors are orthogonal.

(b) (6 points) Is there a time for which the position and velocity are orthogonal? Is so, find all values of t where this occurs.

$$\begin{aligned} \mathbf{r}(t) \cdot \mathbf{v}(t) &= (2\cos 2t)(\sin 2t) + (3)(3t) + (-2\sin 2t)(\cos 2t) \\ &= 2\sin 2t\cos 2t + 9t - 2\sin 2t\cos 2t \\ &= 9t \end{aligned}$$

So $\mathbf{r}(t)$ is orthogonal to $\mathbf{v}(t)$ when $\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$, i.e., when $9t = 0$. Hence, the position and velocity are orthogonal when $t = 0$.

8. (12 points) Below are given 12 equations labeled (a)-(l). Some are in Cartesian coordinates, some in cylindrical coordinates, and some in spherical coordinates. Below are also the graphs of six surfaces, labeled 1-6. Match each equation to its surface graph. (Note: some of the graphs correspond to more than one equation.)

(a) $r = 2\cos\theta$

(b) $z = 2$

(c) $z = r$

(d) $x^2 + y^2 + z^2 = 4$

(e) $(x-1)^2 + y^2 = 1$

(f) $z = e^{y-x}$

(g) $r = 2\sec\theta$

(h) $\rho = 2\sec\phi$

(i) $x = 2$

(j) $\rho = 2$

(k) $z = \sqrt{x^2 + y^2}$

(l) $\phi = \dfrac{\pi}{4}$