Lenarz | Exam #2 | Name: ____________
Math 223 | March 15, 2006 | Section: ____________

Directions: Answer the following questions in the space provided. You may use a calculator and one 3″ × 5″ index card (handwritten, both sides). If you do not show your work, you will receive no credit. The point value of each question is indicated. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!**

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1. Find the limit or show that the limit does not exist:

   (a) (10 points) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy}$

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy} &= \frac{(1)(1)-1}{1+(1)(1)}\\ &= \frac{1-1}{1+1}\\ &= \frac{0}{2}\\ &= 0\end{aligned}$$

   (b) (10 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$

   Along the x-axis ($y=0$), we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2} &= \lim_{(x,0)\to(0,0)} \frac{x}{x^2-(0)^2}\\ &= \lim_{x\to0} \frac{x}{x^2}\\ &= \lim_{x\to0} \frac{1}{x}\\ &\to \infty\end{aligned}$$

   Along the y-axis ($x=0$), we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2} &= \lim_{(0,y)\to(0,0)} \frac{0}{(0)^2-y^2}\\ &= \lim_{y\to0} \frac{0}{-y^2}\\ &= \lim_{x\to0} 0\\ &= 0\end{aligned}$$

   Since the limits are not the same along the two paths, the limit does not exist.

2. Compute all first and second partial derivatives of the following

(a) (10 points) $f(x,y) = e^x \tan y$

$$\begin{aligned}
f_x(x,y) &= e^x \tan y \\
f_y(x,y) &= e^x \sec^2 y \\
f_{xx}(x,y) &= e^x \tan y \\
f_{xy}(x,y) &= e^x \sec^2 y \\
f_{yy}(x,y) &= e^x (2 \sec y)(\sec y \tan y) \\
&= 2e^x \sec^2 y \tan y
\end{aligned}$$

(b) (10 points) $g(x,y,z) = xy + yz + xz$

$$\begin{aligned}
f_x(x,y,z) &= y + z \\
f_y(x,y,z) &= x + z \\
f_z(x,y,z) &= y + x \\
f_{xx}(x,y,z) &= 0 \\
f_{xy}(x,y,z) &= 1 \\
f_{xz}(x,y,z) &= 1 \\
f_{yy}(x,y,z) &= 0 \\
f_{yz}(x,y,z) &= 1 \\
f_{zz}(x,y,z) &= 0
\end{aligned}$$

3. (10 points) Use the chain rule to find $\frac{\partial w}{\partial t}$ if $w = x^2 - 2xy + y^2$, $x = s + t$, and $y = s - t$. Your answer should be in terms of s and t.

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\
&= (2x - 2y)(1) + (-2x + 2y)(-1) \\
&= 2x - 2y + 2x - 2y \\
&= 4x - 4y \\
&= 4(s + t) - 4(s - t) \\
&= 4s + 4t - 4s + 4t \\
&= 8t
\end{aligned}$$

4. Consider the function $f(x, y, z) = xyz$

(a) (10 points) Calculate $\nabla f(x, y, z)$

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle$$

(b) (10 points) Find the directional derivative of $f(x, y, z)$ at the point $(2, 1, 1)$ in the direction of $\mathbf{a} = \langle 2, 1, 2 \rangle$.
We first need a unit vector in the direction of $\mathbf{a}$:

$$\begin{aligned} \mathbf{u} &= \frac{1}{||\mathbf{a}||}\mathbf{a} \\ &= \frac{1}{\sqrt{2^2 + 1^2 + 2^2}} \langle 2, 1, 2 \rangle \\ &= \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle \end{aligned}$$

Then we evaluate $\nabla f(x, y, z)$ at our point:

$$\begin{aligned} \nabla f(2, 1, 1) &= \langle (1)(1), (2)(1), (2)(1) \rangle \\ &= \langle 1, 2, 2 \rangle \end{aligned}$$

So we have

$$\begin{aligned} D_{\mathbf{u}} f(2, 1, 1) &= \mathbf{u} \cdot \nabla f(2, 1, 1) \\ &= \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle \cdot \langle 1, 2, 2 \rangle \\ &= \left(\frac{2}{3}\right)(1) + \left(\frac{1}{3}\right)(2) + \left(\frac{2}{3}\right)(2) \\ &= \frac{2}{3} + \frac{2}{3} + \frac{4}{3} \\ &= \frac{8}{3} \end{aligned}$$

(c) (10 points) What is the maximum rate of change of $f(x, y, z)$ at the point $(2, 1, 1)$? In what direction does it occur?
The maximum rate of change of $f(x, y, z)$ at $(2, 1, 1)$ is given by

$$\begin{aligned} ||\nabla f(2, 1, 1)|| &= ||\langle 1, 2, 2 \rangle|| \\ &= \sqrt{1^2 + 2^2 + 2^2} \\ &= 3 \end{aligned}$$

It occurs in the direction of $\nabla f(2, 1, 1) = \langle 1, 2, 2 \rangle$.

5. (10 points) Find the plane tangent to the hyperboloid

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.
Let $F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$. Then

$$\nabla F(x, y, z) = \langle -4x, -4y, 2z \rangle$$

The normal vector for the plane tangent to the surface at $(1, -1, 4)$ is given by

$$\begin{aligned} \mathbf{n} &= \nabla F(1, -1, 4) \\ &= \langle -4, 4, 8 \rangle \end{aligned}$$

So the equation of the plane is

$$\begin{aligned} -4(x-1) + 4(y-(-1)) + 8(z-4) &= 0 \\ -4x + 4 + 4y + 4 + 8z - 32 &= 0 \\ -4x + 4y + 8z &= 24 \\ -x + y + 2z &= 6 \end{aligned}$$

6. (10 points) Find all critical points of $f(x, y) = 2xy - \frac{1}{2}x^4 - \frac{1}{2}y^4 + 1$. Classify each critical point as a local minimum, local maximum or saddle point.
Notice there are no boundary points to consider. To find our stationary points, we look at

$$\nabla f(x, y) = \langle 2y - 2x^3, 2x - 2y^3 \rangle$$

Stationary points occur when $\nabla f(x, y) = \mathbf{0}$, so we solve the system of equations:

$$\begin{aligned} 2y - 2x^3 &= 0 \\ 2x - 2y^3 &= 0 \end{aligned}$$

which is equivalent to the system

$$\begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned}$$

We can substitute the first equation into the second equation:

$$\begin{aligned} x &= (x^3)^3 \\ x &= x^9 \\ x - x^9 &= 0 \\ x(1 - x^8) &= 0 \\ x(1 - x^4)(1 + x^4) &= 0 \\ x(1 - x^2)(1 + x^2)(1 + x^4) &= 0 \\ x(1 - x)(1 + x)(1 + x^2)(1 + x^4) &= 0 \end{aligned}$$

So we have $x = 0$, $x = 1$ and $x = -1$. If $x = 0$, then $y = 0^3 = 0$, giving us the stationary point $(0, 0)$. If $x = 1$, then $y = (1)^3 = 1$, giving us the stationary point $(1, 1)$. If $x = -1$, then $y = (-1)^3 = -1$, giving us the stationary point $(-1, -1)$. (Notice there are no singular points since ∇f is defined everywhere.)

Now we calculate $D(x, y)$ and apply the second partials test:

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) f_{yy}(x, y) - f_{xy}^2(x, y) \\ &= (-6x^2)(-6y^2) - (2)^2 \\ &= 36x^2y^2 - 4 \end{aligned}$$

Critical Point	$D(x, y)$	$f_{xx}(x, y)$	Conclusion
$(0, 0)$	-4	0	saddle point
$(1, 1)$	32	-6	local maximum
$(-1, -1)$	32	-6	local maximum

7. (**10 BONUS POINTS**) Use Lagrange's method to locate the absolute extrema of $f(x, y) = x^2 + 2y^2 - 2x + 3$ on the set $x^2 + y^2 \leq 10$.

We first look for critical points on the interior of the set, i.e. where $\nabla f = \mathbf{0}$. We have

$$\nabla f(x, y) = \langle 2x - 2, 4y \rangle$$

So we have a critical point when

$$2x - 2 = 0 \qquad \text{AND} \qquad 4y = 0$$

which occurs at the point $1, 0$.

Now we look for extrema on the boundary $x^2 + y^2 = 10$. Our constraint is

$$g(x, y) = x^2 + y^2 - 10$$

First we find

$$\nabla g(x, y) = \langle 2x, 2y \rangle$$

Then set up the system

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned}$$

Which gives us

$$\begin{aligned} 2x - 2 &= 2x\lambda \\ 4y &= 2y\lambda \\ x^2 + y^2 &= 10 \end{aligned}$$

The second equation can be solved

$$\begin{aligned} 4y - 2y\lambda &= 0 \\ 2y(2 - \lambda) &= 0 \end{aligned}$$

So we have either $y = 0$ or $\lambda = 2$.
If $y = 0$, the third equation gives

$$\begin{aligned} x^2 + (0)^2 &= 10 \\ x^2 &= 10 \\ x &= \pm\sqrt{10} \end{aligned}$$

So we have the critical points is $(\sqrt{10}, 0)$ and $(-\sqrt{10}, 0)$.
If $\lambda = 2$, the first equation gives

$$\begin{aligned} 2x - 2 &= 2x(2) \\ 2x - 2 &= 4x \\ -2 &= 2x \\ -1 &= x \end{aligned}$$

Using this value for x in the third equation gives

$$\begin{aligned} (-1)^2 + y^2 &= 10 \\ 1 + y^2 &= 10 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

So we have critical points $(-1, 3)$ and $(-1, -3)$.
Now we check our five points to find the absolute extrema.

$$\begin{aligned} f(1,0) &= 1 + 0 - 2 + 3 \\ &= 2 \\ f(\sqrt{10}, 0) &= 10 + 0 - 2\sqrt{10} + 3 \\ &= 13 - 2\sqrt{10} \\ &\approx 6.6 \\ f(-\sqrt{10}, 0) &= 10 + 0 + 2\sqrt{10} + 3 \\ &= 13 + 2\sqrt{10} \\ &= 19.3 \\ f(-1, 3) &= 1 + 18 + 2 + 3 \\ &= 24 \\ f(-1, -3) &= 1 + 18 + 2 + 3 \\ &= 24 \end{aligned}$$

So we have an absolute minimum value of 2 at the point $(1, 0)$ and an absolute maximum value of 24 at the points $(-1, 3)$ and $(-1, -3)$.