Lenarz	Exam #1	Name: ____________
Math 223	February 8, 2006	Section: ____________

Directions: Answer the following questions in the space provided. You may use a calculator and one 3″ × 5″ index card (handwritten, both sides). If you do not show your work, you will receive no credit. The point value of each question is indicated. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!**

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1. Consider the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 4\mathbf{j} - 3\mathbf{k}$. Calculate

   (a) (5 points) $||\mathbf{u}||$

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{(1)^2 + (-2)^2 + (2)^2} \\ &= \sqrt{1 + 4 + 4} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

   (b) (5 points) $||\mathbf{v}||$

$$\begin{aligned} ||\mathbf{v}|| &= \sqrt{(0)^2 + (4)^2 + (-3)^2} \\ &= \sqrt{0 + 16 + 9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

   (c) (5 points) a unit vector in the direction of $\mathbf{u}$

$$\begin{aligned} \frac{1}{||\mathbf{u}||}\mathbf{u} &= \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \\ &= \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \end{aligned}$$

   (d) (5 points) $\mathbf{u} \cdot \mathbf{v}$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (1)(0) + (-2)(4) + (2)(-3) \\ &= 0 - 8 - 6 \\ &= -14 \end{aligned}$$

   (e) (5 points) the angle between $\mathbf{u}$ and $\mathbf{v}$ (to the nearest degree)

$$
\begin{aligned}
\cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} \\
&= \frac{-14}{(3)(5)} \\
&= -\frac{14}{15} \\
\theta &= \cos^{-1}\left(-\frac{14}{15}\right) \\
&\approx 159^\circ
\end{aligned}
$$

(f) (5 points) the vector projection of $\mathbf{u}$ onto $\mathbf{v}$

$$
\begin{aligned}
\text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v} \\
&= \left(\frac{-14}{5^2}\right)(4\mathbf{j}-3\mathbf{k}) \\
&= -\frac{14}{25}(4\mathbf{j}-3\mathbf{k}) \\
&= -\frac{56}{25}\mathbf{j}+\frac{42}{25}\mathbf{k}
\end{aligned}
$$

2. (12 points) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vectors. State if the expression is a scalar quantity, a vector quantity, or meaningless.

(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ — a. meaningless

(b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ — b. vector

(c) $||\mathbf{a}||(\mathbf{b} \cdot \mathbf{c})$ — c. scalar

(d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ — d. scalar

(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ — e. meaningless

(f) $||\mathbf{a}|| \cdot (\mathbf{b} + \mathbf{c})$ — f. meaningless

(g) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ — g. scalar

(h) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ — h. meaningless

(i) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ — i. vector

(j) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c})$ — j. meaningless

(k) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ — k. meaningless

(l) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ — l. scalar

3. (8 points) Find parametric equations of the line through the points $P(-1,3,2)$ and $Q(2,-1,6)$.
The direction of the line is given by and scalar multiple of the direction vector

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ} \\ &= \langle 2-(-1), -1-3, 6-2\rangle \\ &= \langle 3, -4, 4\rangle\end{aligned}$$

So the parametric equations of the line are

$$\begin{aligned}x &= -1+3t \\ y &= 3-4t \\ z &= 2+4t\end{aligned}$$

Another possible solution is

$$\begin{aligned}x &= 2+3t \\ y &= -1-4t \\ z &= 6+4t\end{aligned}$$

4. (10 points) Find the equation of the plane containing the points $P(-2,1,5)$, $Q(6,2,-3)$, and $R(1,0,1)$.
Find two vectors in the plane

$$\begin{aligned}\mathbf{a} &= \overrightarrow{PQ} \\ &= \langle 6-(-2), 2-1, -3-5\rangle \\ &= \langle 8, 1, -8\rangle \\ \mathbf{b} &= \overrightarrow{PR} \\ &= \langle 1-(-2), 0-1, 1-5\rangle \\ &= \langle 3, -1, -4\rangle\end{aligned}$$

The cross product of these two vectors will be normal to the plane

$$\begin{aligned}\mathbf{n} &= \mathbf{a}\times\mathbf{b} \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 1 & -8 \\ 3 & -1 & -4\end{vmatrix} \\ &= \begin{vmatrix}1 & -8 \\ -1 & -4\end{vmatrix}\mathbf{i} - \begin{vmatrix}8 & -8 \\ 3 & -4\end{vmatrix}\mathbf{j} + \begin{vmatrix}8 & 1 \\ 3 & -1\end{vmatrix}\mathbf{k} \\ &= [(1)(-4)-(-1)(-8)]\mathbf{i} - [(8)(-4)-(-8)(3)]\mathbf{j} + [(8)(-1)-(3)(1)]\mathbf{k} \\ &= [-4-8]\mathbf{i} - [-32-(-24)]\mathbf{j} + [-8-3]\mathbf{k} \\ &= -12\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}\end{aligned}$$

So the equation of the plane is

$$\begin{aligned}(-12)(x-1)+(8)(y-0)+(-11)(z-1) &= 0\\ -12x+12+8y-11z+11 &= 0\\ -12x+8y-11z &= -23\\ 12x-8y+11z &= 23\end{aligned}$$

5. (10 points) Below are given 10 equations labeled (a)-(j). Below are also the graphs of five surfaces, labeled 1-5. Match each equation to its surface graph. (Note: some of the graphs correspond to more than one equation.)

(a) $r = 2\sin\theta$ a. 2

(b) $z = 2$ b. 3

(c) $\sqrt{x^2 + y^2} = z$ c. 1

(d) $x^2 + y^2 + z^2 = 1$ d. 5

(e) $x^2 + (y-1)^2 = 1$ e. 2

(f) $x^2 + y^2 = z$ f. 4

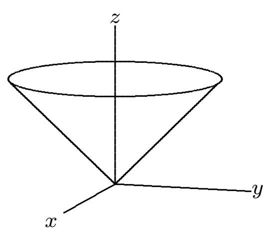
(g) $\rho = 2\sec\phi$ g. 3

(h) $\phi = \dfrac{\pi}{4}$ h. 1

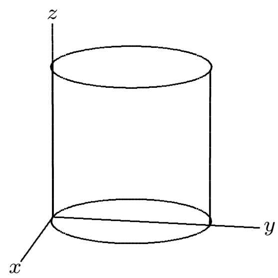
(i) $r^2 = z$ i. 4

(j) $\rho = 1$ j. 5

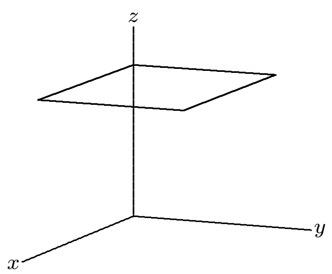
1.



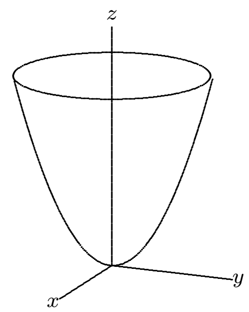
2.



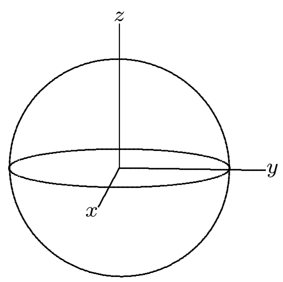
3.



4.



5.



6. (10 points) Convert $\rho = \csc\phi\cot\phi$ from spherical coordinates to

(a) cylindrical coordinates

$$\begin{aligned}
\rho &= \csc\phi\cot\phi \\
\rho &= \frac{\cos\phi}{\sin^2\phi} \\
\rho\sin^2\phi &= \cos\phi \\
\rho^2\sin^2\phi &= \rho\cos\phi \\
(\rho\sin\phi)^2 &= \rho\cos\phi \\
r^2 &= z
\end{aligned}$$

(b) Cartesian coordinates

$$\begin{aligned}
r^2 &= z \\
x^2 + y^2 &= z
\end{aligned}$$

7. (10 points) Find parametric equations for the line tangent to

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$$

at the point $(1, 0, 0)$.

First note that the point $(1, 0, 0)$ corresponds to $t = 0$. The direction vector of the tangent line will be given by $\mathbf{r}'(0)$, so we have

$$\begin{aligned}
\mathbf{r}'(t) &= (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + 2t\mathbf{k} \\
\mathbf{r}'(0) &= (-\sin 0)\mathbf{i} + (\cos 0)\mathbf{j} + 2(0)\mathbf{k} \\
&= \mathbf{j}
\end{aligned}$$

So the parametric equations of the tangent line are

$$\begin{aligned}
x &= 1 \\
y &= t \\
z &= 0
\end{aligned}$$

8. (10 points) Find the velocity and position function of a particle with initial velocity $\mathbf{v}(0) = \mathbf{i}+\mathbf{j}+\mathbf{k}$, initial position $\mathbf{r}(0) = \mathbf{i}-\mathbf{j}-\mathbf{k}$, and acceleration

$$\mathbf{a}(t) = 3t^2\mathbf{i} + 4t^3\mathbf{j} + 2t\mathbf{k}$$

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\ dt \\ &= \int (3t^2\mathbf{i} + 4t^3\mathbf{j} + 2t\mathbf{k})\ dt \\ &= t^3\mathbf{i} + t^4\mathbf{j} + t^2\mathbf{k} + \mathbf{C}\end{aligned}$$

Since $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is the initial velocity we have

$$\begin{aligned}\mathbf{i}+\mathbf{j}+\mathbf{k} &= \mathbf{v}(0) \\ &= (0)^3\mathbf{i} + (0)^4\mathbf{j} + (0)^2\mathbf{k} + \mathbf{C} \\ &= \mathbf{C}\end{aligned}$$

So the constant vector $\mathbf{C} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and we have

$$\mathbf{v}(t) = (t^3+1)\mathbf{i} + (t^4+1)\mathbf{j} + (t^2+1)\mathbf{k}$$

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\ dt \\ &= \int \left((t^3+1)\mathbf{i} + (t^4+1)\mathbf{j} + (t^2+1)\mathbf{k} \right)\ dt \\ &= \left(\frac{1}{4}t^4 + t\right)\mathbf{i} + \left(\frac{1}{5}t^5 + t\right)\mathbf{j} + \left(\frac{1}{3}t^3 + t\right)\mathbf{k} + \mathbf{C}\end{aligned}$$

Since $\mathbf{i}-\mathbf{j}-\mathbf{k}$ is the initial position we have

$$\begin{aligned}\mathbf{i}-\mathbf{j}-\mathbf{k} &= \mathbf{r}(0) \\ &= \left(\frac{1}{4}(0)^4 + 0\right)\mathbf{i} + \left(\frac{1}{5}(0)^5 + 0\right)\mathbf{j} + \left(\frac{1}{3}(0)^3 + 0\right)\mathbf{k} + \mathbf{C} \\ &= \mathbf{C}\end{aligned}$$

So the constant vector $\mathbf{C} = \mathbf{i}-\mathbf{j}-\mathbf{k}$ and we have

$$\mathbf{r}(t) = \left(\frac{1}{4}t^4 + t + 1\right)\mathbf{i} + \left(\frac{1}{5}t^5 + t - 1\right)\mathbf{j} + \left(\frac{1}{3}t^3 + t - 1\right)\mathbf{k}$$

10 POINT BONUS QUESTION

Find the equation of the plane that has intercepts $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

Find two vectors in the plane

$$\begin{aligned}\mathbf{u} &= \langle 0-a, b-0, 0-0\rangle \\ &= \langle -a, b, 0\rangle \\ \mathbf{v} &= \langle 0-a, 0-0, c-0\rangle \\ &= \langle -a, 0, c\rangle\end{aligned}$$

The cross product of these two vectors will be normal to the plane

$$\begin{aligned}\mathbf{n} &= \mathbf{u}\times\mathbf{v} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} \\ &= \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix}\mathbf{i} - \begin{vmatrix} -a & 0 \\ -a & c \end{vmatrix}\mathbf{j} + \begin{vmatrix} -a & b \\ -a & 0 \end{vmatrix}\mathbf{k} \\ &= [(b)(c)-(0)(0)]\mathbf{i} - [(-a)(c)-(-a)(0)]\mathbf{j} + [(-a)(0)-(-a)(b)]\mathbf{k} \\ &= [bc-0]\mathbf{i} - [-ac-0]\mathbf{j} + [0-(-ab)]\mathbf{k} \\ &= bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}\end{aligned}$$

So the equation of the plane is

$$\begin{aligned}(bc)(x-a)+(ac)(y-0)+(ab)(z-0) &= 0 \\ bcx - abc + acy + abz &= 0 \\ bcx + acy + abz &= abc \\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c} &= 1\end{aligned}$$