Lenarz
Math 210

Practice Exam #4
April 23, 2008

Name: ____________________

Directions: Answer the following questions on a separate piece of paper. You may use a calculator. If you do not show your work, you will receive no credit. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Determine if the following are vector spaces. If they are not, explain why.
   (a) the collection of all polynomials of degree 6 with real coefficients with the usual addition and scalar multiplication
   (b) $\left\{ \begin{bmatrix} 1 & a \\ a & -b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ with the usual addition and scalar multiplication

2. Determine if the following sets are subspaces of the vector space $\mathbb{R}^n$ (with the usual vector addition and scalar multiplication)
   (a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = 2t, y = -3t, z = 8t \right\}$
   (b) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = 7t - 1, y = 2 - 3t, z = t \right\}$
   (c) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x + y = 1 \right\}$

3. Show that the set of all symmetric $n \times n$ matrices is a subspace of $M_{n,n}$.

4. Show that the set of all 2×2 matrices A such that $\det A = 0$ is not a subspace of $M_{2,2}$.

5. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 2y + 5z = 0 \right\}$ be a plane in $\mathbb{R}^3$.
   (a) Show that $W = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right)$
   (b) Show that $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set.

6. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in a vector space V. Show that $\{\mathbf{x} + \mathbf{y}, \mathbf{x} + 2\mathbf{y}, \mathbf{x} + 3\mathbf{y}\}$ is a linearly dependent set.

7. Determine whether the following are linearly independent:
   (a) $\{1+2x, 3-7x, 55-x\}$
   (b) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

8. Find a basis and the dimension for the following vector spaces:
   (a) the symmetric $n \times n$ matrices
   (b) the skew-symmetric $n \times n$ matrices
   (c) $\{a + bx - ax^2 : a, b \in \mathbb{R}\}$

9. Find a basis for the vector space $V = \{A \in M_{2,4} : a_{23} = a_{14}\}$

10. Determine whether T is a linear transformation
    (a) $T : M_{n,n} \to M_{n,n}$ defined by $T(A) = A^T A$
    (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{y}^T\mathbf{x}\mathbf{y}$ where $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

11. If $T : \mathcal{P}_2 \to M_{2,2}$ is a linear transformation so that $T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T(1+x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T(1+x+x^2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find $T(c + bx + ax^2)$.

12. Find the rank and nullity of $T : M_{3,3} \to M_{3,3}$ defined by $T(A) = A - A^T$.

13. If $T : U \to V$ and $S : V \to W$ are linear transformations so that $\text{range}(T) \subseteq \ker(S)$, what can be deduced about $S \circ T$?

14. Indicate TRUE if true in all cases or FALSE otherwise. Justify your answer.
    (a) The zero vector is an element of every vector space.
    (b) The zero vector is an element of every subspace of a vector space.
    (c) In any vector space, $c_1\mathbf{u} = c_2\mathbf{u}$ implies that $c_1 = c_2$.
    (d) In any vector space, $c\mathbf{u} = c\mathbf{v}$ implies that $\mathbf{u} = \mathbf{v}$.
    (e) The empty set is a subspace of every vector space.
    (f) If $\dim V = n$, then any set of more than n vectors in V must be linearly dependent.
    (g) A subset of a linearly independent set of vectors is linearly independent.
    (h) Every vector space has a finite basis.
    (i) If V is a vector space and $\dim V = n$ then any subspace W of V must satisfy $\dim W < n$.
    (j) If $T : V \to W$ is a linear transformation and $\ker(T) = V$, then $W = \{\mathbf{0}\}$.

Proofs

15. Let A and B be nonzero $n \times n$ matrices such that A is symmetric and B is skew-symmetric. Prove that $\{A, B\}$ is linearly independent.

16. Let $T : V \to V$ be a linear transformation so that $T \circ T = T$. Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \mathbf{v}$ or $T(\mathbf{v}) = \mathbf{0}$.

17. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors in a vector space V with the property that every vector in V can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$ in exactly one way. Prove that S is a basis for V.

18. If U and W are vector spaces, define $U \times W = \{(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in U, \mathbf{w} \in W\}$ where addition and scalar multiplication in $U \times W$ are performed componentwise, that is

    $$(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2)$$

    and

    $$k(\mathbf{u}, \mathbf{w}) = (k\mathbf{u}, k\mathbf{w})$$

    Prove that $U \times W$ is a vector space.

19. Let U and W be subspaces of a finite dimensional vector space V. Define $T : U \times W \to V$ by $T(\mathbf{u}, \mathbf{w}) = \mathbf{u} - \mathbf{w}$. (See previous question for a definition of $U \times W$.) Prove T is a linear transformation.