

**Directions:** Answer the following questions on a separate piece of paper. You may use a calculator. If you do not show your work, you will receive no credit. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Find all eigenvalues and associated eigenspaces for the following matrices:

(a)  $\begin{bmatrix} 1 & -2 \\ 2 & 6 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix}$

(c)  $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 5 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

2. Determine if  $\lambda = 1$  is an eigenvalue of

$$\begin{bmatrix} 2 & 3 & 1 & 2 \\ 4 & 4 & 7 & 2 \\ 4 & 1 & 6 & 6 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

3. The matrix

$$A = \begin{bmatrix} 7 & 0 & -8 \\ 0 & -1 & 0 \\ 4 & 0 & -5 \end{bmatrix}$$

has eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

What are the eigenvalues of  $A$  associated with these eigenvectors?

4. Use expansion by minors to find the determinant of the following:

(a)  $\begin{bmatrix} 3 & 0 & 0 & -2 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 & -3 \\ -4 & 0 & 1 & 0 & 6 \\ 0 & -1 & 0 & 3 & 2 \end{bmatrix}$

(b)  $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix}$

5. Let  $A$  be a  $4 \times 4$  matrix with  $\det A = -3$ . Compute the following determinants:

- (a)  $\det A^T$
- (b)  $\det A^{-1}$
- (c)  $\det A^3$
- (d)  $\det(2A)$
- (e)  $\det(3A)^2$

6. Suppose  $A, B$  and  $C$  are all  $3 \times 3$  matrices such that  $\det A = 2$ ,  $\det B = 5$  and  $\det C = 7$ . Find

- (a)  $\det(3AB)$
- (b)  $\det(A^T BC^{-1})$

7. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & x & 5 \\ -2 & -2 & 2 \end{bmatrix}$$

- (a) Find  $\det A$ .
- (b) For what values of  $x$  is  $A$  invertible?
- (c) For those  $x$  such that  $A^{-1}$  exists find  $\det A^{-1}$ .

8. If  $A$  and  $B$  are  $n \times n$  matrices with  $A \sim B$ , which of the following must be true?

- (a)  $A^2 \sim B^2$
- (b)  $A^T \sim B^T$
- (c)  $B \sim A$
- (d)  $AB \sim BA$

9. For each matrix, determine if it is diagonalizable. If it is, find a diagonal matrix  $D$  and an invertible matrix  $P$  so that  $P^{-1}AP = D$ .

(a)  $A = \begin{bmatrix} -4 & 0 & 4 \\ -5 & 1 & 4 \\ -2 & 0 & 2 \end{bmatrix}$

(b)  $A = \begin{bmatrix} 2 & 0 & 0 \\ -5 & 17 & -20 \\ -5 & 15 & -18 \end{bmatrix}$

(c)  $A = \begin{bmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

10. Let  $A$  be a  $4 \times 4$  matrix with eigenvalues 5, 2 and  $-2$ . If the eigenspace for  $\lambda = 2$  is two-dimensional, can we conclude that  $A$  is diagonalizable? Explain.

11. Let  $A$  be a  $7 \times 7$  matrix with three eigenvalues. If one eigenspace is two-dimensional and the second is three-dimensional, can we conclude that  $A$  is diagonalizable? Explain.

12. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 2 \end{bmatrix}$$

(a) Show that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  form an orthogonal set.

(b) If  $V$  is the subspace of  $\mathbb{R}^4$  spanned by  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$ , find an orthonormal basis for  $V$ .

(c)  $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ \frac{9}{2} \\ \frac{1}{2} \end{bmatrix}$  is a vector in the subspace  $V$ . Write  $\mathbf{w}$  as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$ .

13. Is

$$\left\{ \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \\ -\frac{2\sqrt{2}}{3} \end{bmatrix} \right\}$$

an orthonormal basis for  $\mathbb{R}^3$ ?

14. Find the orthogonal complement  $W^\perp$  of

$$W = \text{span} \left( \begin{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \end{pmatrix} \right)$$

15. Find the orthogonal decomposition of  $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$  with respect to

$$W = \text{span} \left( \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix} \end{pmatrix} \right)$$

16. Indicate TRUE if true in all cases or FALSE otherwise. Justify your answer.
- (a) If an  $n \times n$  matrix  $A$  has two identical rows, then  $\det A = 0$ .
  - (b) The determinant of an upper triangular matrix equals the product of its diagonal entries.
  - (c) For an invertible matrix  $A$ ,  $\det A^T = \frac{1}{\det A}$
  - (d) If  $A$  is an  $n \times n$  matrix and  $A\mathbf{x} = \mathbf{0}$  for some  $\mathbf{x} \neq \mathbf{0}$ , then  $\det A = 0$ .
  - (e) If  $A$  is a matrix with positive entries, then  $\det A > 0$ .
  - (f) If  $A$  and  $B$  are  $n \times n$  invertible matrices, then  $\det(B^{-1}AB^T) = \det A$ .
  - (g) The geometric multiplicity of an eigenvalue is greater than or equal to its algebraic multiplicity.
  - (h) Every  $n \times n$  matrix has  $n$  distinct eigenvalues.
  - (i) Similar matrices have the same eigenvalues.
  - (j) Similar matrices have the same eigenvectors.

## Proofs

17. If  $P$  is an invertible matrix and  $A = P^{-1}BP$  and  $\mathbf{v}$  is an eigenvector for  $A$  corresponding to the eigenvalue  $\lambda$ , show that  $P\mathbf{v}$  is an eigenvector for  $B$  corresponding to  $\lambda$ .
18. Prove that if  $A \sim B$ , then  $A^n \sim B^n$  for positive integers  $n$ .
19. An  $n \times n$  matrix  $N$  is said to be normal if  $NN^T = N^TN$ . Prove that an orthogonal matrix is normal.