4. Compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, and (d) the algebraic and geometric multiplicity of each eigenvalue if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

(a)

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + (1)\begin{vmatrix} 0 & 1-\lambda \\ 1 & 1 \end{vmatrix} \\
&= (1-\lambda)\Big[(1-\lambda)(-\lambda) - (1)(1)\Big] + \Big[(0)(1) - (1-\lambda)(1)\Big] \\
&= (1-\lambda)(\lambda^2 - \lambda - 1) - (1-\lambda) \\
&= (1-\lambda)\Big[(\lambda^2 - \lambda - 1) - 1\Big] \\
&= (1-\lambda)(\lambda^2 - \lambda - 2) \\
&= (1-\lambda)(\lambda - 2)(\lambda + 1)
\end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, so the eigenvalues of A are $\lambda = 1, 2, -1$.

(c) For $\lambda = 1$: find the basis for the null space of $A - I$.

$$\begin{aligned}
[A - I|\mathbf{0}] &= \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \\
&\xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = 1$ must have $x_1 + x_2 = 0$ and $x_3 = 0$, that is $x_2 = -x_1$ and $x_3 = 0$, so we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

thus

$$E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$$

For $\lambda = 2$: find the basis for the null space of $A - 2I$.

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = 2$ must have $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$, that is $x_1 = x_3$ and $x_2 = x_3$, so we have

$$\mathbf{x} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

thus

$$E_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

For $\lambda = -1$: find the basis for the null space of $A + I$.

$$[A + I|\mathbf{0}] = \left[\begin{array}{rrr|r} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = -1$ must have $x_1 + \frac{1}{2}x_3 = 0$ and $x_2 + \frac{1}{2}x_3 = 0$, that is $x_1 = -\frac{1}{2}x_3$ and $x_2 = -\frac{1}{2}x_3$, so we have

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

thus

$$E_{-1} = \text{span}\left\{\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right\}$$

(d) .

λ	Algebraic	Geometric
1	1	1
2	1	1
-1	1	1

8. Compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, and (d) the algebraic and geometric multiplicity of each eigenvalue if

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Solution:

(a)

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & -1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)\Big[(1-\lambda)(1-\lambda) - (-1)(-1)\Big] \\ &= (2-\lambda)(\lambda^2 - 2\lambda) \\ &= -\lambda(2-\lambda)^2 \end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, so the eigenvalues of A are $\lambda = 0, 2$.

(c) For $\lambda = 0$: find the basis for the null space of $A - 0I = A$.

$$[A|\mathbf{0}] = \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = 0$ must have $x_1 - x_3 = 0$ and $x_2 = 0$, that is $x_3 = x_1$ and $x_2 = 0$, so we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

thus

$$E_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 2$: find the basis for the null space of $A - 2I$.

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{rrr|r} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = 2$ must have $x_1 + x_2 + x_3 = 0$, that is $x_1 = -x_2 - x_3$, so we have

$$\mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

thus

$$E_2 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) .

λ	Algebraic	Geometric
0	1	1
2	2	2

10. Compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, and (d) the algebraic and geometric multiplicity of each eigenvalue if

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution:

(a)

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 1 & 1 & 0 \\ 0 & 1-\lambda & 4 & 5 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda)\Big[(3-\lambda)(2-\lambda) - (0)(1)\Big] \\ &= (2-\lambda)^2(1-\lambda)(3-\lambda) \end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, so the eigenvalues of A are $\lambda = 1, 2, 3$.

(c) For $\lambda = 1$: find the basis for the null space of $A - I$.

$$[A - I|\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ of A associated with $\lambda = 1$ must have $x_1 + x_2 = 0$, $x_3 = 0$ and $x_4 = 0$, that is $x_2 = -x_1$, $x_3 = 0$ and $x_4 = 0$, so we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

thus

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = 2$: find the basis for the null space of $A - 2I$.

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = 2$ must have $x_2 - x_4 = 0$ and $x_3 + x_4 = 0$, that is $x_2 = x_4$ and $x_3 = -x_4$, so we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

thus

$$E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 3$: find the basis for the null space of $A - 3I$.

$$[A - 3I|\mathbf{0}] = \left[\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So any eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A associated with $\lambda = 3$ must have $x_1 - 3x_3 = 0$, $x_2 - 2x_3 = 0$, and $x_4 = 0$, that is $x_1 = 3x_3$, $x_2 = 2x_3$, and $x_4 = 0$, so we have

$$\mathbf{x} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

thus

$$E_3 = \text{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(d) .

λ	Algebraic	Geometric
1	1	1
2	2	2
3	1	1

20. Let A be a nilpotent matrix (that is, $A^m = O$ for some $m > 1$). Show that $\lambda = 0$ is the only eigenvalue of A.

Solution: Suppose A is a nilpotent matrix so that $A^m = O$ where $m > 1$ and λ is an eigenvalue of A with eigenvector $\mathbf{x} \neq \mathbf{0}$.

$$\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} && \text{(def. of eigenvalue/eigenvector)}\\
A^m\mathbf{x} &= \lambda^m\mathbf{x} && \text{(Theorem in class)}\\
O\mathbf{x} &= \lambda^m\mathbf{x} && (A^m = O)\\
\mathbf{0} &= \lambda^m\mathbf{x} &&\\
0 &= \lambda^m && (\text{since } \mathbf{x} \neq \mathbf{0})\\
0 &= \lambda &&
\end{aligned}$$

☒

21. Let A be an idempotent matrix (that is, $A^2 = A$). Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of A.

Solution: Suppose λ is an eigenvalue of A with eigenvector $\mathbf{v} \neq \mathbf{0}$. Then

$$\begin{aligned}
\lambda\mathbf{v} &= A\mathbf{v} && \text{(def. of eigenvalue)}\\
&= A^2\mathbf{v} && (A = A^2)\\
&= \lambda^2\mathbf{v} && \text{(Theorem 4.18)}\\
\mathbf{0} &= \lambda^2\mathbf{v} - \lambda\mathbf{v} &&\\
&= (\lambda^2 - \lambda)\mathbf{v} && \text{(distributive prop.)}\\
&= \lambda(\lambda - 1)\mathbf{v} &&
\end{aligned}$$

Since $\mathbf{v} \neq \mathbf{0}$, either $\lambda = 0$ or $\lambda = 1$. ☒

22. If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

Solution: Suppose λ is an eigenvalue of A with eigenvector $\mathbf{v} \neq \mathbf{0}$.

$$\begin{aligned}
(A - cI)\mathbf{v} &= A\mathbf{v} - (cI)\mathbf{v} && \text{(distributive prop.)}\\
&= \lambda\mathbf{v} - c\mathbf{v} && (A\mathbf{v} = \lambda\mathbf{v}, I\mathbf{v} = \mathbf{v})\\
&= (\lambda - c)\mathbf{v} && \text{(distributive prop.)}
\end{aligned}$$

So $\lambda - c$ is an eigenvalue of $A - cI$ with eigenvector $\mathbf{v}$. ☒

26. Find the companion matrix of $p(x) = x^2 - 7x + 12$ and then find the characteristic polynomial of $C(p)$.

Solution: The companion matrix for $p(x) = x^2 - 7x + 12$ is

$$C(p) = \begin{bmatrix} 7 & -12 \\ 1 & 0 \end{bmatrix}$$

and the characteristic polynomial of $C(p)$ is

$$\begin{aligned} \det(C(p) - \lambda I) &= \begin{vmatrix} 7-\lambda & -12 \\ 1 & -\lambda \end{vmatrix} \\ &= (7-\lambda)(-\lambda) - (-12)(1) \\ &= \lambda^2 - 7\lambda + 12 \end{aligned}$$

28. (a) Show that the companion matrix $C(p)$ of $p(x) = x^2 + ax + b$ has characteristic polynomial $\lambda^2 + a\lambda + b$.

Solution:

$$C(p) = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix}$$

so the characteristic polynomial is

$$\begin{aligned} \det|A - \lambda I| &= \begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} \\ &= (-a-\lambda)(-\lambda) - (-b)(1) \\ &= \lambda^2 + a\lambda + b \end{aligned}$$

(b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

Solution: Suppose λ is an eigenvalue of $C(p)$. Then it will be a root of the characteristic polynomial, i.e.

$$\begin{aligned} \lambda^2 + a\lambda + b &= 0 \\ \lambda^2 &= -a\lambda - b \end{aligned}$$

If $\mathbf{v} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$, then

$$\begin{aligned} C(p)\mathbf{v} &= \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -a\lambda - b \\ \lambda \end{bmatrix} \\ &= \begin{bmatrix} \lambda^2 \\ \lambda \end{bmatrix} \\ &= \lambda \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \end{aligned}$$

So $\mathbf{v}$ is an eigenvector of $C(p)$ associated with λ. ⊠