HW # 13 §3.2

Lenarz - Math 210
Solutions
Due: February 15, 2008

6. Write (if possible) $B = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$ as a linear combination of

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution: We are trying to find c_1, c_2, and c_3 so that

$$c_1A_1 + c_2A_2 + c_3A_3 = B$$

Thus we have

$$\begin{aligned} B &= c_1A_1 + c_2A_2 + c_3A_3 \\ \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} &= c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1 + c_3 & -c_2 + c_3 \\ c_2 & c_1 + c_3 \end{bmatrix} \end{aligned}$$

Hence we have the system of equations (in the variables c_1, c_2, and c_3)

$$\begin{array}{rcrcrcr} c_1 & & & + & c_3 & = & 2 \\ & - & c_2 & + & c_3 & = & 3 \\ & & c_2 & & & = & -4 \\ c_1 & & & + & c_3 & = & 2 \end{array}$$

Which has augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we have $c_1 = 3, c_2 = -4$, and $c_3 = -1$ and hence

$$3A_1 - 4A_2 - A_3 = B$$

10. Find the general form of span(A_1, A_2, A_3) where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution: span(A_1, A_2, A_3) is the set of all possible linear combinations of A_1, A_2, and A_3. So an arbitrary matrix $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is in span(A_1, A_2, A_3) if there are scalars c_1, c_2, and c_3 so that

$$B = c_1A_1 + c_2A_2 + c_3A_3$$

that is,

$$\begin{aligned} \begin{bmatrix} w & x \\ y & z \end{bmatrix} &= c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1 + c_3 & -c_2 + c_3 \\ c_2 & c_1 + c_3 \end{bmatrix} \end{aligned}$$

Hence we have the system of equations (in the variables c_1, c_2, and c_3)

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & w \\ & - & c_2 & + & c_3 & = & x \\ & & c_2 & & & = & y \\ c_1 & & & + & c_3 & = & z \end{array}$$

Which has augmented matrix

$$\left[\begin{array}{rrr|c} 1 & 0 & 1 & w \\ 0 & -1 & 1 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & w - x - y \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x + y \\ 0 & 0 & 0 & z - w \end{array}\right]$$

So in order for the system to be consistent, we must have $z - w = 0$, that is $z = w$. So the general form for a matrix in span(A_1, A_2, A_3) is

$$\begin{bmatrix} w & x \\ y & w \end{bmatrix}$$

where w, x, and y are real numbers.

14. Determine if

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

are linearly independent.

Solution: Suppose

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then we have

$$\begin{bmatrix} c_1 + 2c_2 + c_3 & c_1 + c_2 + 2c_3 \\ c_1 - c_2 + 4c_3 & c_1 + 3c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence we have the homogeneous system of equations (in the variables c_1, c_2, and c_3)

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & c_3 & = & 0 \\ c_1 & + & c_2 & + & 2c_3 & = & 0 \\ c_1 & - & c_2 & & 4c_3 & = & 0 \\ c_1 & & & + & 3c_3 & = & 0 \end{array}$$

Which has augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since c_3 is a free variable, there are an infinite number of solutions and thus a nontrivial linear combination of our matrices that sum to the zero matrix, meaning that the matrices are linearly dependent.

24. Find conditions on a, b, c, and d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Solution: We want to find a, b, c, and d so that $AB = BA$.

$$\begin{aligned} AB &= BA \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{bmatrix} a-c & b-d \\ -a+c & -b+d \end{bmatrix} &= \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} \end{aligned}$$

Hence we have the homogeneous system of equations (in the variables a, b, c, and d)

$$\begin{array}{rrrrrrrcl} & & b & - & c & & & = & 0 \\ a & & & & & - & d & = & 0 \\ -a & & & & & + & d & = & 0 \\ & - & b & + & c & & & = & 0 \end{array}$$

Which has augmented matrix

$$\left[\begin{array}{rrrr|r} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have $a = d$ and $b = c$, so any matrix B that commutes with A has form

$$B = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

26. Find conditions on a, b, c, and d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We want to find a, b, c, and d so that $A_1 B = BA_1$ and $A_2 B = BA_2$.

$$\begin{aligned} A_1 B &= BA_1 \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \end{aligned}$$

Hence we have the system of equations (in the variables a, b, c, and d)

$$\begin{aligned} a &= a \\ b &= 0 \\ c &= 0 \\ 0 &= 0 \end{aligned}$$

Thus we have $b = 0$ and $c = 0$.

$$\begin{aligned} A_2 B &= BA_2 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} &= \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \end{aligned}$$

Hence we have the system of equations (in the variables a, b, c, and d)

$$\begin{aligned} 0 &= 0 \\ b &= 0 \\ c &= 0 \\ d &= d \end{aligned}$$

Thus we have $b = 0$ and $c = 0$. So any matrix B that commutes with both A_1 and A_2 has form

$$B = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

33. Using induction, prove that for all $n \geq 1$, $(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$

Solution: Base step: $n = 1$

$$(A_1)^T = A_1^T \checkmark$$

Induction step: Assume that

$$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T \qquad \text{(Induction Hypothesis)}$$

Then we have

$$\begin{aligned}(A_1 A_2 \cdots A_n A_{n+1})^T &= \Big((A_1 A_2 \cdots A_n) A_{n+1}\Big)^T && \text{(associativity)}\\ &= A_{n+1}^T (A_1 A_2 \cdots A_n)^T && \text{(property of transpose)}\\ &= A_{n+1}^T A_n^T \cdots A_2^T A_1^T && \text{(induction hypothesis)}\end{aligned}$$

☒

35. (a) Prove that if A and B are symmetric $n \times n$ matrices, then so is $A + B$.

Solution: Suppose that A and B are symmetric $n \times n$ matrices. By the definition of symmetric, $A^T = A$ and $B^T = B$. Also,

$$\begin{aligned}(A+B)^T &= A^T + B^T && \text{(property of transpose)}\\ &= A + B && (A = A^T \text{ and } B = B^T)\end{aligned}$$

So $(A+B)^T = A+B$, and by the definition of symmetric, $A+B$ is symmetric. ☒

(b) Prove that if A is a symmetric $n \times n$ matrix, then so is kA for any scalar k.

Solution: Suppose that A is a symmetric $n \times n$ matrix. By the definition of symmetric, $A^T = A$. Also,

$$\begin{aligned}(kA)^T &= kA^T && \text{(property of transpose)}\\ &= kA && (A = A^T)\end{aligned}$$

So $(kA)^T = kA$, and by the definition of symmetric, kA is symmetric. ☒

40. Prove that if A and B are skew-symmetric $n \times n$ matrices, then so is $A + B$.

Solution: Suppose that A and B are skew-symmetric matrices. By the definition of skew-symmetric, $A^T = -A$ and $B^T = -B$. Also,

$$\begin{aligned}(A+B)^T &= A^T + B^T && \text{(property of transpose)}\\ &= -A + (-B) && (A^T = -A \text{ and } B^T = -B)\\ &= -(A+B)\end{aligned}$$

So $(A + B)^T = -(A + B)$, and by the definition of skew-symmetric, $A + B$ is skew-symmetric. ☒

42. Prove that if A is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric.

Solution: Suppose A is an $n \times n$ matrix. Then

$$\begin{aligned}(A - A^T)^T &= A^T - (A^T)^T && \text{(property of transpose)}\\ &= A^T - A && \text{(property of transpose)}\\ &= -(A - A^T)\end{aligned}$$

So by definition, $A - A^T$ is skew-symmetric. ☒

43. (a) Prove that any square matrix A can be written as a sum of a symmetric matrix and a skew-symmetric matrix. (*Hint:* Consider Theorem 3.5 and Exercise 42.)

Solution: Suppose A is a matrix. Let $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$. Then

$$\begin{aligned}B + C &= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\\ &= \frac{1}{2}(A + A^T + A - A^T)\\ &= A\end{aligned}$$

Also,

$$\begin{aligned}B^T &= \left(\tfrac{1}{2}(A + A^T)\right)^T\\ &= \tfrac{1}{2}(A + A^T)^T && \text{(property of transpose)}\\ &= \tfrac{1}{2}(A + A^T) && \text{(Theorem 3.5: } (A + A^T) \text{ is symmetric)}\\ &= B\end{aligned}$$

so B is symmetric and

$$\begin{aligned} C^T &= \left(\tfrac{1}{2}(A - A^T)\right)^T \\ &= \tfrac{1}{2}(A - A^T)^T && \text{(property of transpose)} \\ &= \tfrac{1}{2}(-(A - A^T)) && \text{(part(b): } (A - A^T) \text{ is skew-symmetric)} \\ &= -C \end{aligned}$$

so C is skew-symmetric. ☒

(b) Illustrate part (a) for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: Let $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$. Then

$$\begin{aligned} B &= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T\right) \\ &= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
C &= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T\right) \\
&= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}
\end{aligned}$$

Note that $B^T = B$ and $C^T = -C$ and $A = B + C$.