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Math 210

Practice Exam #3
March 28, 2007

Name: ____________________

**Directions:** Answer the following questions on a separate piece of paper. You may use a calculator. If you do not show your work, you will receive no credit. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Find all eigenvalues and associated eigenspaces for the following matrices:

   (a) $\begin{bmatrix} 1 & -2 \\ 2 & 6 \end{bmatrix}$

**Solution:** We find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -2 \\ 2 & 6-\lambda \end{vmatrix} \\
&= (1-\lambda)(6-\lambda) - (2)(-2) \\
&= (6 - 7\lambda + \lambda^2) + 4 \\
&= \lambda^2 - 7\lambda + 10 \\
&= (\lambda - 5)(\lambda - 2)
\end{aligned}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$.
For $\lambda_1 = 2$, we have

$$\begin{aligned}
A - 2I &= \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix} \\
&\xrightarrow{rref} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

So we know that

$$x_1 + 2x_2 = 0$$

Which is equivalent to

$$x_1 = -2x_2$$

Letting $x_2 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_2$.
For $\lambda_2 = 5$, we have

$$\begin{aligned}
A - 5I &= \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix} \\
&\xrightarrow{rref} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

So we know that

$$x_1 + \frac{1}{2}x_2 = 0$$

Which is equivalent to

$$x_1 = -\frac{1}{2}x_2$$

Letting $x_2 = 2s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} -s \\ 2s \end{bmatrix} = s\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$ is a basis for $E_5$.

(b) $\begin{bmatrix} 1 & 8 \\ 1 & -1 \end{bmatrix}$

**Solution:** We find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix} \\
&= (1-\lambda)(-1-\lambda) - (8)(1) \\
&= (-1+\lambda^2) - 8 \\
&= \lambda^2 - 9 \\
&= (\lambda - 3)(\lambda + 3)
\end{aligned}$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$.
For $\lambda_1 = 3$, we have

$$\begin{aligned}
A - 3I &= \begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix} \\
&\xrightarrow{rref} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

So we know that

$$x_1 - 4x_2 = 0$$

Which is equivalent to

$$x_1 = 4x_2$$

Letting $x_2 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} 4s \\ s \end{bmatrix} = s\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

So $\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$ is a basis for $E_3$.
For $\lambda_2 = -3$, we have

$$\begin{aligned} A + 3I &= \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$x_1 + 2x_2 = 0$$

Which is equivalent to

$$x_1 = -2x_2$$

Letting $x_2 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ is a basis for $E_{-3}$.

(c) $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 5 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

**Solution:** We find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 0 & 1 \\ 2 & 5-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix} \\ &= (3-\lambda)\Big((3-\lambda)(5-\lambda) - (2)(0)\Big) \\ &= (3-\lambda)^2(5-\lambda) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$.

For $\lambda_1 = 3$, we have

$$\begin{aligned} A - 3I &= \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Which is equivalent to

$$\begin{aligned} x_1 &= -x_2 \\ x_3 &= 0 \end{aligned}$$

Letting $x_2 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $E_3$.

For $\lambda_2 = 5$, we have

$$\begin{aligned} A - 5I &= \begin{bmatrix} -2 & 0 & 1 \\ 2 & 0 & 5 \\ 0 & 0 & -2 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$\begin{aligned} x_1 &= 0 \\ x_3 &= 0 \end{aligned}$$

Letting $x_2 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $E_5$.

2. Determine if $\lambda = 1$ is an eigenvalue of

$$\begin{bmatrix} 2 & 3 & 1 & 2 \\ 4 & 4 & 7 & 2 \\ 4 & 1 & 6 & 6 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

**Solution:** For $\lambda = 1$ to be an eigenvalue of $A$, we must have $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$. So we solve the system

$$\begin{aligned} (A - I)\mathbf{x} &= \mathbf{0} \\ \begin{bmatrix} 1 & 3 & 1 & 2 \\ 4 & 3 & 7 & 2 \\ 4 & 1 & 5 & 6 \\ 1 & 3 & 1 & 2 \end{bmatrix} \mathbf{x} &= \mathbf{0} \end{aligned}$$

The coefficient matrix has rref

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so we have a non-trivial solution $\mathbf{x} \neq \mathbf{0}$ and thus $\lambda = 1$ is an eigenvalue of $A$.

3. The matrix

$$A = \begin{bmatrix} 7 & 0 & -8 \\ 0 & -1 & 0 \\ 4 & 0 & -5 \end{bmatrix}$$

has eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

What are the eigenvalues of $A$?

**Solution:** For $\mathbf{v}_1$ to be an eigenvector of $A$ we must have $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$, so we multiply

$$\begin{aligned} A\mathbf{v}_1 &= \begin{bmatrix} 7 & 0 & -8 \\ 0 & -1 & 0 \\ 4 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \\ &= -\mathbf{v}_1 \end{aligned}$$

So $\lambda_1 = -1$ is an eigenvalue of $A$ with eigenvector $\mathbf{v}_1$.
For $\mathbf{v}_2$ to be an eigenvector of $A$ we must have $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$, so we multiply

$$\begin{aligned} A\mathbf{v}_2 &= \begin{bmatrix} 7 & 0 & -8 \\ 0 & -1 & 0 \\ 4 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \\ &= -\mathbf{v}_2 \end{aligned}$$

So $\lambda_2 = -1$ is an eigenvalue of $A$ with eigenvector $\mathbf{v}_2$.
For $\mathbf{v}_3$ to be an eigenvector of $A$ we must have $A\mathbf{v}_3 = \lambda_3 \mathbf{v}_3$, so we multiply

$$\begin{aligned} A\mathbf{v}_3 &= \begin{bmatrix} 7 & 0 & -8 \\ 0 & -1 & 0 \\ 4 & 0 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} \\ &= 3\mathbf{v}_2 \end{aligned}$$

So $\lambda_3 = 3$ is an eigenvalue of $A$ with eigenvector $\mathbf{v}_3$.

4. Use expansion by minors to find the determinant of the following:

(a) $\begin{bmatrix} 3 & 0 & 0 & -2 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 & -3 \\ -4 & 0 & 1 & 0 & 6 \\ 0 & -1 & 0 & 3 & 2 \end{bmatrix}$

**Solution:** Expand on the third column to get

$$\begin{vmatrix} 3 & 0 & 0 & -2 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 & -3 \\ -4 & 0 & 1 & 0 & 6 \\ 0 & -1 & 0 & 3 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 3 & 0 & -2 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 5 & -3 \\ 0 & -1 & 3 & 2 \end{vmatrix}$$

Now expand on the second row

$$(-1)\begin{vmatrix} 3 & 0 & -2 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 5 & -3 \\ 0 & -1 & 3 & 2 \end{vmatrix} = (-1)(2)\begin{vmatrix} 3 & -2 & 4 \\ 0 & 5 & -3 \\ 0 & 3 & 2 \end{vmatrix}$$

Expand again (first column)

$$(-1)(2)\begin{vmatrix} 3 & -2 & 4 \\ 0 & 5 & -3 \\ 0 & 3 & 2 \end{vmatrix} = (-1)(2)(3)\begin{vmatrix} 5 & -3 \\ 3 & 2 \end{vmatrix}$$

Now evaluate the $2 \times 2$ determinant

$$(-1)(2)(3)\begin{vmatrix} 5 & -3 \\ 3 & 2 \end{vmatrix} = -6(10+9) = -114$$

(b) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix}$

**Solution:** Expand on the first row to get

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 0 & 0 & 3 \\ 2 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{vmatrix}$$

Now expand on the first row

$$(-1)\begin{vmatrix} 0 & 0 & 0 & 3 \\ 2 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{vmatrix} = (-1)(-3)\begin{vmatrix} 2 & 0 & 4 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

Expand again (second column)

$$(-1)(-3)\begin{vmatrix} 2 & 0 & 4 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{vmatrix} = (-1)(-3)(5)\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}$$

Now evaluate the $2 \times 2$ determinant

$$(-1)(-3)(5)\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} = 15(-2-4) = -90$$

5. Let $A$ be a $4 \times 4$ matrix with $\det A = -3$. Compute the following determinants:

(a) $\det A^T$

**Solution:** Since $\det A^T = \det A$, we have

$$\det A^T = -3$$

(b) $\det A^{-1}$

**Solution:** Since $\det A^{-1} = \frac{1}{\det A}$, we have

$$\det A^{-1} = -\frac{1}{3}$$

(c) $\det A^3$

**Solution:** Since $\det A^k = (\det A)^k$, we have

$$\det A^3 = (-3)^3 = -27$$

(d) $\det(2A)$

**Solution:** Since $\det(kA) = k^n \det A$ if $A$ is $n \times n$, we have

$$\det(2A) = 2^4(-3) = -48$$

(e) $\det(3A)^2$

**Solution:** We have

$$\begin{aligned}\det(kA)^p &= (\det(kA))^p \\ &= (k^n \det A)^p\end{aligned}$$

if $A$ is $n \times n$, so

$$\det(3A)^2 = (3^4(-3))^2 = 59049$$

6. Suppose $A$, $B$ and $C$ are all $3 \times 3$ matrices such that $\det A = 2$, $\det B = 5$ and $\det C = 7$. Find

(a) $\det(3AB)$

**Solution:** Notice that

$$\begin{aligned}\det(kAB) &= k^n \det(AB) \\ &= k^n \det A \det B\end{aligned}$$

if $AB$ is $n \times n$, so we have

$$\det(3AB) = 3^3(2)(5) = 270$$

(b) $\det(A^T BC^{-1})$

**Solution:** Notice that

$$\begin{aligned}\det(A^T BC^{-1}) &= \det A^T \det B \det C^{-1} \\ &= \det A \det B \frac{1}{\det C} \\ &= \frac{\det A \det B}{\det C}\end{aligned}$$

so we have

$$\det(A^T BC^{-1}) = \frac{(2)(5)}{7} = \frac{10}{7}$$

7. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & x & 5 \\ -2 & -2 & 2 \end{bmatrix}$$

(a) Find $\det A$.

**Solution:** Expansion on the first column yields

$$\begin{vmatrix} 1 & 3 & -1 \\ 0 & x & 5 \\ -2 & -2 & 2 \end{vmatrix} = (1)\begin{vmatrix} x & 5 \\ -2 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -1 \\ x & 5 \end{vmatrix}$$
$$= (2x+10) - 2(15+x)$$
$$= -20$$

(b) For what values of $x$ is $A$ invertible?

**Solution:** $A$ is invertible if $\det A \neq 0$. Since $\det A = -20$ does not depend on $x$, $A$ is invertible for all real $x$.

(c) For those $x$ such that $A^{-1}$ exists find $\det A^{-1}$.

**Solution:** Since $\det A^{-1} = \frac{1}{\det A}$, we have

$$\det A^{-1} = -\frac{1}{20}$$

8. If $A$ and $B$ are $n \times n$ matrices with $A \sim B$, which of the following must be true?

(a) $A^2 \sim B^2$

**Solution:** If $A \sim B$, then there exists an invertible matrix $P$ so that $P^{-1}AP = B$. Notice that

$$B^2 = (P^{-1}AP)^2$$
$$= (P^{-1}AP)(P^{-1}AP)$$
$$= P^{-1}AIAP$$
$$= P^{-1}A^2P$$

so $A^2 \sim B^2$.

(b) $A^T \sim B^T$

**Solution:** If $A \sim B$, then there exists an invertible matrix $P$ so that $P^{-1}AP = B$, or equivalently $AP = PB$. Notice that

$$\begin{aligned} (AP)^T &= (PB)^T \\ P^T A^T &= B^T P^T \\ A^T(P^T)^{-1} &= (P^T)^{-1}B^T \end{aligned}$$

so $A^T \sim B^T$.

(c) $B \sim A$

**Solution:** This is Theorem 4.21b.

(d) $AB \sim BA$

**Solution:** Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So if $AB \sim BA$ then there would exist an invertible matrix $P$ so that

$$\begin{aligned} P^{-1}(AB)P &= BA \\ P^{-1}(AB)P &= O \\ PP^{-1}(AB)P &= PO \\ I(AB)P &= O \\ (AB)P &= O \\ (AB)PP^{-1} &= OP^{-1} \\ (AB)I &= O \\ AB &= O \end{aligned}$$

But $AB \neq O$, so they cannot be similar.

9. For each matrix, determine if it is diagonalizable. If it is, find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$.

   (a) $A = \begin{bmatrix} -4 & 0 & 4 \\ -5 & 1 & 4 \\ -2 & 0 & 2 \end{bmatrix}$

**Solution:** We first find the eigenvalues and associated eigenspaces of the matrix. We find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -4-\lambda & 0 & 4 \\ -5 & 1-\lambda & 4 \\ -2 & 0 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} -4-\lambda & 4 \\ -2 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\Big((-4-\lambda)(2-\lambda) - (4)(-2)\Big) \\ &= (1-\lambda)(-8 + 2\lambda + \lambda^2 + 8) \\ &= (1-\lambda)(\lambda^2 + 2\lambda) \\ &= \lambda(1-\lambda)(\lambda+2) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = -2$.
For $\lambda_1 = 0$, we have

$$\begin{aligned} A - 0I &= A \\ &= \begin{bmatrix} -4 & 0 & 4 \\ -5 & 1 & 4 \\ -2 & 0 & 2 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Which is equivalent to

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \end{aligned}$$

Letting $x_3 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_0$.
For $\lambda_2 = 1$, we have

$$\begin{aligned} A - I &= \begin{bmatrix} -5 & 0 & 4 \\ -5 & 0 & 4 \\ -2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$\begin{aligned} x_1 &= 0 \\ x_3 &= 0 \end{aligned}$$

Letting $x_2 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $E_1$.
For $\lambda_3 = -2$, we have

$$\begin{aligned} A + 2I &= \begin{bmatrix} -2 & 0 & 4 \\ -5 & 3 & 4 \\ -2 & 0 & 4 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

Which is equivalent to

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= 2x_3 \end{aligned}$$

Letting $x_3 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} 2s \\ 2s \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_{-2}$.

For each eigenvalue, the geometric multiplicity is the same as its algebraic multiplicity, so the matrix is diagonalizable and $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & 0 & 0 \\ -5 & 17 & -20 \\ -5 & 15 & -18 \end{bmatrix}$

**Solution:** We first find the eigenvalues and associated eigenspaces of the matrix. We find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ -5 & 17-\lambda & -20 \\ -5 & 15 & -18-\lambda \end{vmatrix} \\ &= (2-\lambda) \begin{vmatrix} 17-\lambda & -20 \\ 15 & -18-\lambda \end{vmatrix} \\ &= (2-\lambda)\Big((17-\lambda)(-18-\lambda) - (15)(-20)\Big) \\ &= (2-\lambda)(-306 + \lambda + \lambda^2 + 300) \\ &= (2-\lambda)(\lambda^2 + \lambda - 6) \\ &= (2-\lambda)(\lambda + 3)(\lambda - 2) \end{aligned}$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -3$.

For $\lambda_1 = 2$, we have

$$\begin{aligned} A - 2I &= \begin{bmatrix} 0 & 0 & 0 \\ -5 & 15 & -20 \\ -5 & 15 & -20 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$x_1 - 3x_2 + 4x_3 = 0$$

Which is equivalent to

$$x_1 = 3x_2 - 4x_3$$

Letting $x_2 = s \neq 0$ and $x_3 = t \neq 0$ we have

$$\mathbf{x} = \begin{bmatrix} 3s - 4t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_2$.

For $\lambda_2 = -3$, we have

$$\begin{aligned} A + 3I &= \begin{bmatrix} 5 & 0 & 0 \\ -5 & 20 & -20 \\ -5 & 15 & -15 \end{bmatrix} \\ &\xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So we know that

$$\begin{aligned} x_1 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Which is equivalent to

$$\begin{aligned} x_1 &= 0 \\ x_2 &= x_3 \end{aligned}$$

Letting $x_3 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis for $E_{-3}$.

For each eigenvalue, the geometric multiplicity is the same as its algebraic multiplicity, so the matrix is diagonalizable and $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 3 & -4 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

**Solution:** We first find the eigenvalues and associated eigenspaces of the matrix. We find the eigenvalues by finding the roots of the characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} \\
&= (-1)\begin{vmatrix} -1 & 2 \\ -3-\lambda & 3 \end{vmatrix} + (-2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} \\
&= (-1)\Big((-1)(3) - (2)(-3-\lambda)\Big) + (-2-\lambda)\Big((2-\lambda)(-3-\lambda) - (5)(-1)\Big) \\
&= (-1)(-3+6+2\lambda) + (-2-\lambda)(-6+\lambda+\lambda^2+5) \\
&= (-1)(3+2\lambda) + (-2-\lambda)(\lambda^2+\lambda-1) \\
&= -3-2\lambda-2\lambda^2-2\lambda+2-\lambda^3-\lambda^2+\lambda \\
&= -\lambda^3-3\lambda^2-3\lambda-1 \\
&= -(\lambda^3+3\lambda^2+3\lambda+1) \\
&= -(\lambda+1)^3
\end{aligned}$$

So the eigenvalue is $\lambda_1 = -1$.

For $\lambda_1 = -1$, we have

$$\begin{aligned}
A + I &= \begin{bmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{bmatrix} \\
&\xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

So we know that

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Which is equivalent to

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \end{aligned}$$

Letting $x_3 = s \neq 0$, we have

$$\mathbf{x} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $E_{-1}$.

The geometric multiplicity of $\lambda_1 = 1$, but the algebraic multiplicity of $\lambda_1 = 3$, so the matrix is not diagonalizable.

10. Let $A$ be a $4 \times 4$ matrix with eigenvalues $5, 2$ and $-2$. If the eigenspace for $\lambda = 2$ is two-dimensional, can we conclude that $A$ is diagonalizable? Explain.

**Solution:** Yes. Since we have a $4 \times 4$ matrix, the characteristic polynomial is a fourth degree polynomial. Since we only have three eigenvalues, one of them must have algebraic multiplicity two. We know that the algebraic multiplicity is greater than or equal to the geometric multiplicity, so since the geometric multiplicity of $\lambda = 2$ is 2, we know its algebraic multiplicity is also 2. The other two eigenvalues have algebraic and geometric multiplicity one, so the matrix is diagonalizable.

11. Let $A$ be a $7 \times 7$ matrix with three eigenvalues. If one eigenspace is two-dimensional and the second is three-dimensional, can we conclude that $A$ is diagonalizable? Explain.

**Solution:** No. Since we have a $7 \times 7$ matrix, the characteristic polynomial is a seventh degree polynomial. Since we only have three eigenvalues, at least one of them must have algebraic multiplicity two or more. We know that the algebraic multiplicity is greater than or equal to the geometric multiplicity, so since the geometric multiplicity of $\lambda_1$ is 2, we know its algebraic multiplicity is 2 or greater. The geometric multiplicity of $\lambda_2$ is 3, so we know its algebraic multiplicity is 3 or greater. For $A$ to be diagonalizable, we need the geometric and algebraic multiplicity to be the same, so we need $\lambda_1$ to have algebraic multiplicity 2 and $\lambda_2$ to have algebraic multiplicity 3. This leaves $\lambda_3$ with algebraic multiplicity 2, (since the sum of all the algebraic multiplicities has to be seven), but $\lambda_3$ can have geometric multiplicity 1 which means $A$ is not guaranteed to be diagonalizable.

12. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 2 \end{bmatrix}$$

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form an orthogonal set.

**Solution:**

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= (1)(0) + (0)(2) + (1)(1) + (1)(-1) \\ &= 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= (1)(0) + (0)(2) + (1)(-2) + (1)(2) \\ &= 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= (0)(0) + (2)(2) + (1)(-2) + (-1)(2) \\ &= 0 \end{aligned}$$

Since every pair of vectors in the set is orthogonal, the set is orthogonal.

(b) If $V$is the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, find an orthonormal basis for $V$.

**Solution:** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms an orthogonal basis for $V$. To make it orthonormal, we just need to normalize the vectors. So $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for $V$ if

$$\begin{aligned}
\mathbf{q}_1 &= \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 \\
&= \frac{1}{\sqrt{3}}\mathbf{v}_1 \\
&= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \\
\mathbf{q}_2 &= \frac{1}{||\mathbf{v}_2||}\mathbf{v}_2 \\
&= \frac{1}{\sqrt{6}}\mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \\
\mathbf{q}_3 &= \frac{1}{||\mathbf{v}_3||}\mathbf{v}_3 \\
&= \frac{1}{2\sqrt{3}}\mathbf{v}_3 \\
&= \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}
\end{aligned}$$

(c) $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ \frac{9}{2} \\ \frac{1}{2} \end{bmatrix}$ is a vector in the subspace $V$. Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

**Solution:** Using Theorem 5.2 we have

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

where

$$c_1 = \frac{\mathbf{w}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1} = \frac{7}{3}$$

$$c_2 = \frac{\mathbf{w} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} = \frac{2}{6} = \frac{1}{3}$$
$$c_3 = \frac{\mathbf{w} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} = \frac{-10}{12} = -\frac{5}{6}$$

⊠

13. Is

$$\left\{ \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \\ -\frac{2\sqrt{2}}{3} \end{bmatrix} \right\}$$

an orthonormal basis for $\mathbb{R}^3$?

**Solution:**

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 &= \left(\frac{2}{3}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{2}{3}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{3}\right)(0) \\
&= 0 \\
\mathbf{v}_1 \cdot \mathbf{v}_3 &= \left(\frac{2}{3}\right)\left(\frac{\sqrt{2}}{6}\right) + \left(\frac{2}{3}\right)\left(\frac{\sqrt{2}}{6}\right) + \left(\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right) \\
&= 0 \\
\mathbf{v}_2 \cdot \mathbf{v}_3 &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{6}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{6}\right) + (0)\left(-\frac{2\sqrt{2}}{3}\right) \\
&= 0
\end{aligned}$$

Since every pair of vectors in the set is orthogonal, the set is orthogonal.

It is also easy to check that each vector is a unit vector:

$$\begin{aligned}
||\mathbf{v}_1|| &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \\
&= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} \\
&= 1 \\
||\mathbf{v}_2|| &= \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + (0)^2} \\
&= \sqrt{\frac{2}{4} + \frac{2}{4} + 0} \\
&= 1 \\
||\mathbf{v}_3|| &= \sqrt{\left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2 + \left(-\frac{2\sqrt{2}}{3}\right)^2} \\
&= \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} \\
&= 1
\end{aligned}$$

Since this is an orthogonal set of unit vectors, it is an orthonormal set.

14. Find the orthogonal complement $W^\perp$ of

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}\right)$$

**Solution:** $W = \text{col}(A)$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

so $W^\perp = (\text{col}(A))^\perp = \text{null}(A^T)$ Note that

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \end{bmatrix}$$

so any vector in $W^\perp = \text{null}(A^T)$ has form

$$\begin{bmatrix} -x_3 - \frac{3}{2}x_4 \\ \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

and a basis for $W^\perp = \text{null}(A^T)$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

15. Find the orthogonal decomposition of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ with respect to

$$W = \text{span}\left( \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

**Solution:** We can take

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

where

$$\mathbf{w} = \text{proj}_W(\mathbf{v}) \text{ and } \mathbf{w}^\perp = \text{perp}_W(\mathbf{v})$$

Notice that

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

forms an orthogonal basis for $W$, so

$$
\begin{aligned}
\mathbf{w} &= \operatorname{proj}_W(\mathbf{v}) \\
&= \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}) + \operatorname{proj}_{\mathbf{u}_2}(\mathbf{v}) + \operatorname{proj}_{\mathbf{u}_3}(\mathbf{v}) \\
&= \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 + \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}}{\mathbf{u}_3 \cdot \mathbf{u}_3}\right)\mathbf{u}_3 \\
&= \left(\frac{1}{3}\right)\mathbf{u}_1 + \left(\frac{-2}{3}\right)\mathbf{u}_2 + \left(\frac{7}{15}\right)\mathbf{u}_3 \\
&= \begin{bmatrix} \frac{11}{15} \\ \frac{4}{5} \\ -\frac{19}{15} \\ \frac{22}{15} \end{bmatrix} \\
\mathbf{w}^{\perp} &= \operatorname{perp}_W(\mathbf{v}) \\
&= \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) \\
&= \begin{bmatrix} \frac{4}{15} \\ -\frac{4}{5} \\ \frac{4}{15} \\ \frac{8}{15} \end{bmatrix}
\end{aligned}
$$

16. Indicate TRUE if true in all cases or FALSE otherwise. Justify your answer.

(a) If an $n \times n$ matrix $A$ has two identical rows, then $\det A = 0$.

**Solution:** True. This is Theorem 4.3c.

(b) The determinant of an upper triangular matrix equals the product of its diagonal entries.

**Solution:** True. This is Theorem 4.2.

(c) For an invertible matrix $A$, $\det A^T = \frac{1}{\det A}$

**Solution:** False. $\det A^T = \det A$ and $\det A^{-1} = \frac{1}{\det A}$.

(d) If $A$ is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\det A = 0$.

**Solution:** True. By the Fundamental Theorem of Invertible matrices, $A\mathbf{x} = \mathbf{0}$ iff $A$ is not invertible iff $\det A = 0$.

(e) If $A$ is a matrix with positive entries, then $\det A > 0$.

**Solution:** False. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then $\det A = 1 - 4 = -3 < 0$.

(f) If $A$ and $B$ are $n \times n$ invertible matrices, then $\det(B^{-1}AB^T) = \det A$.

**Solution:** True.

$$\begin{aligned} \det(B^{-1}AB^T) &= \det B^{-1} \det A \det B^T \\ &= \frac{1}{\det B} \det A \det B \\ &= \det A \end{aligned}$$

(g) The geometric multiplicity of an eigenvalue is greater than or equal to its algebraic multiplicity.

**Solution:** False. By Lemma 4.26, The geometric multiplicity of an eigenvalue is *less* than or equal to its algebraic multiplicity.

(h) Every $n \times n$ matrix has $n$ distinct eigenvalues.

**Solution:** False. Every $n \times n$ matrix has $n$ eigenvalues, but they can be repeated (i.e., have algebraic multiplicity greater than 1).

(i) Similar matrices have the same eigenvalues.

**Solution:** True. This is Theorem 4.22e.

(j) Similar matrices have the same eigenvectors.

**Solution:** False. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Notice $A \sim B$ since $P^{-1}AP = B$ when

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Both matrices have eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$. The eigenspaces for $A$ are $E_0 = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $E_1 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. The eigenspaces for $B$ are $E_0 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and $E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

# Proofs

17. If $P$ is an invertible matrix and $A = P^{-1}BP$ and $\mathbf{v}$ is an eigenvector for $A$ corresponding to the eigenvalue $\lambda$, show that $P\mathbf{v}$ is an eigenvector for $B$ corresponding to $\lambda$.

**Solution:** Suppose $\mathbf{v}$ is an eigenvector for $A$ corresponding to the eigenvalue $\lambda$, $P$ is an invertible matrix and $A = P^{-1}BP$. Then $PAP^{-1} = B$ and $A\mathbf{v} = \lambda\mathbf{v}$. So we have

$$\begin{aligned} B(P\mathbf{v}) &= (PAP^{-1})(P\mathbf{v}) && (B = PAP^{-1}) \\ &= PAI\mathbf{v} \\ &= PA\mathbf{v} \\ &= P(\lambda\mathbf{v}) && (A\mathbf{v} = \lambda\mathbf{v}) \\ &= \lambda(P\mathbf{v}) && \text{(property of scalar mult.)} \end{aligned}$$

So $P\mathbf{v}$ is an eigenvector of $B$ associated with the eigenvalue $\lambda$. ⊠

18. Let $A$ be an idempotent matrix (that is, $A^2 = A$). Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of $A$.

**Solution:** Suppose $\lambda$ is an eigenvalue of $A$ with eigenvector $\mathbf{v} \neq \mathbf{0}$. Then

$$\begin{aligned} \lambda\mathbf{v} &= A\mathbf{v} && \text{(def. of eigenvalue)} \\ &= A^2\mathbf{v} && (A = A^2) \\ &= \lambda^2\mathbf{v} && \text{(Theorem 4.18)} \\ \mathbf{0} &= \lambda^2\mathbf{v} - \lambda\mathbf{v} \\ &= (\lambda^2 - \lambda)\mathbf{v} && \text{(distributive prop.)} \\ &= \lambda(\lambda - 1)\mathbf{v} \end{aligned}$$

Since $\mathbf{v} \neq \mathbf{0}$, either $\lambda = 0$ or $\lambda = 1$. ⊠

19. Prove that if $A \sim B$, then $A^n \sim B^n$ for positive integers $n$.

**Solution:** Suppose $A \sim B$. Then there exists an invertible matrix $P$ so that $P^{-1}AP = B$. So we have

$$\begin{aligned} B^n &= (P^{-1}AP)^n \\ &= \underbrace{(P^{-1}AP)(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{n \text{ times}} \\ &= P^{-1}\underbrace{AA\cdots A}_{n \text{ times}}P \\ &= P^{-1}A^nP \end{aligned}$$

So $A^n \sim B^n$. ⊠

20. Prove that if $A$ is a diagonalizable matrix such that every eigenvalue of $A$ is either 0 or 1, then $A$ is idempotent (that is $A^2 = A$).

**Solution:** Suppose $A$ is diagonalizable. Then there exists an invertible matrix $P$ and a diagonal matrix $D$ so that $A = PDP^{-1}$. $A$ and $D$ have the same eigenvalues and the eigenvalues occur on the diagonal of $D$. Since all of the eigenvalues of $A$ are 0 or 1, all of the entries on the diagonal of $D$ are either 0 or 1. $D^2$ is the diagonal matrix found by squaring the entries on the diagonal of $D$. Since in this case the entries on the diagonal are 0 and 1, the square of the entries are also 0 and 1. Thus $D^2 = D$.

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= (PDP^{-1})^2 \\ &= (PDP^{-1})(PDP-1) \\ &= PDIDP^{-1} \\ &= PD^2P^{-1} \\ &= PDP^{-1} && (D^2 = D) \\ &= A \end{aligned}$$

⊠

21. An $n \times n$ matrix $N$ is said to be normal if $NN^T = N^TN$. Prove that an orthogonal matrix is normal.

**Solution:** Let $Q$ be an orthogonal matrix. Then by Theorem 5.5, $Q^T = Q^{-1}$. So we have

$$\begin{aligned} QQ^T &= QQ^{-1} \\ &= I \\ &= Q^{-1}Q \\ &= Q^TQ \end{aligned}$$

So $Q$ is normal. ⊠