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Math 210

Practice Exam #2
October 29, 2007

Name: ____________________

Directions: Answer the following questions on a separate piece of paper. You may use a calculator. If you do not show your work, you will receive no credit. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Perform the indicated matrix operation or state "undefined".

(a) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -3 \\ 4 & -2 & 6 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -3 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 2+1 & -3+3 & 1+(-3) \\ 0+4 & 5+(-2) & 2+6 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 8 \end{bmatrix}$$

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 2 \end{bmatrix}$

Solution: This is undefined. Two matrices must be the same size in order to add them.

(c) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 4 & -2 & 6 \end{bmatrix}$

Solution: This is undefined. The number of columns of the first matrix is not the same as the number of rows of the second matrix.

(d) $\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 2 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} (2)(3)+(-1)(2) & (2)(-1)+(-1)(4) & (2)(5)+(-1)(2) \\ (1)(3)+(5)(2) & (1)(-1)+(5)(4) & (1)(5)+(5)(2) \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -6 & 8 \\ 13 & 19 & 15 \end{bmatrix}$$

(e) $\begin{bmatrix} -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$

Solution:

$$\begin{aligned} \begin{bmatrix} -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} &= (-1)(4) + (3)(2) + (2)(-3) \\ &= -4 \end{aligned}$$

(f) $\begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \end{bmatrix}$

Solution:

$$\begin{aligned} \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \end{bmatrix} &= \begin{bmatrix} (4)(-1) & (4)(3) & (4)(2) \\ (2)(-1) & (2)(3) & (2)(2) \\ (-3)(-1) & (-3)(3) & (-3)(2) \end{bmatrix} \\ &= \begin{bmatrix} -4 & 12 & 8 \\ -2 & 6 & 4 \\ 3 & -9 & -6 \end{bmatrix} \end{aligned}$$

(g) $-3 \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 4 & 3 \end{bmatrix}$

Solution:

$$\begin{aligned} -3 \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 4 & 3 \end{bmatrix} &= \begin{bmatrix} -3(-1) & -3(2) \\ -3(0) & -3(1) \\ -3(4) & -3(3) \end{bmatrix} \\ &= -3 \begin{bmatrix} 3 & -6 \\ 0 & -3 \\ -12 & -9 \end{bmatrix} \end{aligned}$$

(h) $\begin{bmatrix} 2 & 5 \\ -1 & 6 \\ 4 & 3 \end{bmatrix}^T$

Solution:

$$\begin{bmatrix} 2 & 5 \\ -1 & 6 \\ 4 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 6 & 3 \end{bmatrix}$$

2. Let A, B, and X be square matrices, I the identity matrix of the same size, and assume that A and B are invertible. Solve the following matrix equation for X:

$$(AB)^{-1}XA + 2I = A$$

Solution:

$$\begin{aligned}
(AB)^{-1}XA + 2I &= A \\
(AB)^{-1}XA &= A - 2I \\
(AB)(AB)^{-1}XA &= (AB)(A - 2I) \\
IXA &= (AB)A - (AB)(2I) \\
XA &= ABA - 2AB \\
XAA^{-1} &= (ABA - 2AB)A^{-1} \\
XI &= ABAA^{-1} - 2ABA^{-1} \\
X &= AB - 2ABA^{-1}
\end{aligned}$$

3. If A is a square matrix such that $A^3 = O$, show that $(I - A)^{-1} = I + A + A^2$.

Solution:

$$\begin{aligned}
(I - A)(I + A + A^2) &= I(I + A + A^2) - A(I + A + A^2) && \text{(distributivity)} \\
&= II + IA + IA^2 - AI - AA - AA^2 && \text{(distributivity)} \\
&= I + A + A^2 - A - A^2 - A^3 \\
&= I - A^3 \\
&= I - O && \text{(since } A^3 = O\text{)} \\
&= I
\end{aligned}$$

Thus $(I - A)^{-1} = I + A + A^2$ by Theorem 3.13.

4. Find A^{-1} for each of the following matrices:

(a)

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Solution: Using the Gauss-Jordan method we have

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: Using the Gauss-Jordan method we have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

5. Which of the following are subspaces of $\mathbb{R}^2$? Explain.

   (a) The set of all $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ such that $x^2 + y^2 + 1 = 0$.

   Solution: This is not a subspace because it does not contain the zero vector. If $x = y = 0$, then $0^2 + 0^2 + 1 = 1 \neq 0$.

   (b) The set of all $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ such that $x^2 + y^2 = 0$.

   Solution: Notice that this set is $\{\mathbf{0}\}$, which is a (trivial) subspace of $\mathbb{R}^2$.

6. Determine whether or not each of the following is a subspace of $\mathbb{R}^3$. If not, explain. If so, find a basis for the subspace.

   (a) The set of all vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $z = x - y$.

   Solution: Notice this is the plane $x - y - z = 0$, which goes through the origin, so it is a subspace. All vectors in the subspace are of the form

$$\begin{bmatrix} x \\ y \\ x-y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

   so the subspace is spanned by the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

   The set is linearly independent since

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0} \Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_1 - c_2 \end{bmatrix} = \mathbf{0} \Leftrightarrow c_1 = c_2 = 0$$

   so the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

   is a basis for the plane $x - y - z = 0$.

(b) The set of all vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x \geq 0$.

Solution: This is not a subspace because it is not closed under scalar multiplication. For example, if $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$ (which is in the set) and $c = -1$, then $c\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -5 \end{bmatrix}$, which is not in the set.

(c) The set of all vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

Solution: The set is the null space of the matrix, so it is a subspace. To find a basis, we look at the rref of the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we have x_3 as a free variable. Note that

$$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$

and

$$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$$

So letting $x_3 = t$, we have the solution

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for the null space.

(d) The set of all vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution: This is not a subspace because it does not contain the zero vector.

7. Let

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 1 & -2 \end{bmatrix}$$

(a) Find a basis for row(A).

(b) Find a basis for col(A).

(c) Find a basis for null(A).

(d) What are the rank and nullity of A?

Solution:

(a) Note that

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 1 & -2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for row(A) is the set of nonzero rows of rref(A), i.e.,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

(b) Using the same rref as in (a), we look at the columns in A that correspond to the columns in rref(A) with leading 1s. So a basis for col(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) Note that

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 0 \\ 2 & 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So we have two free variables x_3 and x_4. Note that

$$x_1 + x_3 - 2x_4 = 0 \Rightarrow x_1 = -x_3 + 2x_4$$

and

$$x_2 - x_3 + 2x_4 = 0 \Rightarrow x_2 = x_3 - 2x_4$$

So letting $x_3 = s$ and $x_4 = t$, we have the solution

$$\begin{bmatrix} -s + 2t \\ s - 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

So

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the null space.

(d) From parts (a) and (b), we can see that rank(A) $= 2$. From part (c), we have nullity(A) $= 2$.

8. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & b \\ 1 & 0 & 1 & 4 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

Determine a basis for row(A) and col(A). Your answers should depend on b.

Solution: We begin by finding the rref of A.

$$\begin{bmatrix} 0 & 1 & 1 & b \\ 1 & 0 & 1 & 4 \\ 1 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & b \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & b \\ 0 & -1 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & b-3 \end{bmatrix}$$

Now if $b = 3$, we have

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so a basis for row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

and a basis for col(A) is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

If $b \neq 3$, then we have

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & b-3 \end{bmatrix} \xrightarrow{\frac{1}{b-3} R_3} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - bR_3]{R_1 - 4R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So a basis for row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and a basis for col(A) is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b \\ 4 \\ 1 \end{bmatrix} \right\}$$

9. Find the nullity of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 1 & 0 & 7 \\ 1 & 3 & 1 & 4 \end{bmatrix}$$

Solution: Notice that

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 1 & 0 & 7 \\ 1 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

so $\text{rank}(A) = 3$. By the rank theorem we have

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns of A, so we have

$$\begin{aligned} 3 + \text{nullity}(A) &= 4 \\ \text{nullity}(A) &= 1 \end{aligned}$$

10. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the mapping given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 0 \\ x+2y \end{bmatrix}$. Show that T is a linear transformation.

Solution:

$$\begin{aligned} T\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -y \\ 0 \\ x+2y \end{bmatrix} \\ &= x\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

So T is a matrix transformation, and hence a linear transformation, by Theorem 3.30.

11. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the mapping given by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2 \end{bmatrix}$. Is T is a linear transformation? Why or why not?

Solution: This is not a linear transformation because if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then

$$T(\mathbf{u}+\mathbf{v}) = T\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ 2 \end{bmatrix}$$

but

$$T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix} u_1 \\ 2 \end{bmatrix} + \begin{bmatrix} v_1 \\ 2 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ 4 \end{bmatrix}$$

so the first property for linear transformations is not satisfied.

12. If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation such that

$$T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } T\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

find $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution: We will try to find a way to write $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Then we can use the properties of linear transformations to find $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\begin{aligned} c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2c_1 + 2c_2 \\ 3c_1 + 4c_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

So we have the system with augmented matrix

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

So we have $c_1 = -1$ and $c_2 = 1$, i.e.

$$-\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus

$$\begin{aligned} T\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= T\left(-\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) \\ &= -T\begin{bmatrix} 2 \\ 3 \end{bmatrix} + T\begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &= -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

13. If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that

$$T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \text{ and } T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

find $T\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Solution: We will try to find a way to write $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then we can use the properties of linear transformations to find $T\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$$\begin{aligned} c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ \begin{bmatrix} -c_1 + c_2 \\ c_1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

So we have the system with augmented matrix

$$\left[\begin{array}{cc|c} -1 & 1 & 3 \\ 1 & 0 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

So we have $c_1 = 2$ and $c_2 = 5$, i.e.

$$2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Thus

$$\begin{aligned} T\begin{bmatrix} 3 \\ 2 \end{bmatrix} &= T\left(2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= 2T\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 5\begin{bmatrix} -1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -5 \end{bmatrix} \end{aligned}$$

14. Indicate TRUE if true in all cases or FALSE otherwise. Justify your answer.

(a) If A and B are both symmetric matrices, then AB is symmetric.

Solution: False. If

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

A and B are both symmetric ($A^T = A$ and $B^T = B$), but AB is not symmetric.

(b) If A^{-1} and B^{-1} both exist and commute, then A and B commute.

Solution: True. Suppose A^{-1} and B^{-1} both exist and commute. Then we have

$$\begin{aligned} A^{-1}B^{-1} &= B^{-1}A^{-1} \\ (BA)^{-1} &= (AB)^{-1} \qquad \text{(property of inverse)} \\ \left((BA)^{-1}\right)^{-1} &= \left((AB)^{-1}\right)^{-1} \\ BA &= AB \end{aligned}$$

(c) If A is an $n \times n$ matrix and $AB = O$, then $B = O$.

Solution: False. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then $AB = O$ but $B \neq O$.

(d) The rank of an $m \times n$ matrix is at most n.

Solution: True. Look at the different cases. If $m \le n$ then the rank is at most the number of rows of the matrix, which is m and $m \le n$.
If $m > n$ (more rows than columns), then the maximum number of nonzero rows in the rref is the number of columns, which is n.

(e) The $m \times n$ zero matrix is the only $m \times n$ matrix with rank zero.

Solution: True. The only matrix that has the zero matrix as its ref is the zero matrix.

(f) If S is a subspace of $\mathbb{R}^n$ and $\dim S = n$, then $S = \mathbb{R}^n$.

Solution: True. If $\dim S = n$, then there is a set of n vectors in $\mathbb{R}^n$ that are linearly independent and span S. But any set of n vectors in $\mathbb{R}^n$ that are linearly independent span $\mathbb{R}^n$.

(g) If A is an $m \times n$ matrix and N is the set of all solutions to $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, then N is a subspace of $\mathbb{R}^n$.

Solution: False. N is not closed under addition. If $\mathbf{x}$ and $\mathbf{y}$ are in N, then

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= A\mathbf{x} + A\mathbf{y} \\ &= \mathbf{b} + \mathbf{b} \\ &= 2\mathbf{b} \\ &\neq \mathbf{b} \end{aligned}$$

So $\mathbf{x}+\mathbf{y}$ is not in N.

(h) A linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ always maps a straight line to a parallel straight line.

Solution: False. It can map a straight line to a point.

(i) A linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ always maps a set of parallel lines to a set of parallel lines.

Solution: False. It can map a set of parallel lines to a single line, to two points, or a single point.

Proofs

15. Prove that if A and B are symmetric $n \times n$ matrices, then so is $A + B$.

Solution: Suppose that A and B are symmetric $n \times n$ matrices. By the definition of symmetric, $A^T = A$ and $B^T = B$. Also,

$$\begin{aligned} (A+B)^T &= A^T + B^T && \text{(property of transpose)} \\ &= A + B && (A = A^T \text{ and } B = B^T) \end{aligned}$$

So $(A+B)^T = A + B$, and by the definition of symmetric, $A + B$ is symmetric. ⊠

16. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation and suppose that $\mathbf{v}$ is a vector such that $T(\mathbf{v}) \neq \mathbf{0}$ but $T^2(\mathbf{v}) = \mathbf{0}$ (where $T^2 = T \circ T$). Prove that $\mathbf{v}$ and $T(\mathbf{v})$ are linearly independent.

Solution: Suppose that $\mathbf{v}$ is a vector such that $T(\mathbf{v}) \neq \mathbf{0}$ but $T^2(\mathbf{v}) = \mathbf{0}$ (where $T^2 = T \circ T$). Notice that if

$$c_1\mathbf{v} + c_2T(\mathbf{v}) = \mathbf{0}$$

then

$$\begin{aligned} T\Big(c_1\mathbf{v} + c_2T(\mathbf{v})\Big) &= T(\mathbf{0}) \\ c_1T(\mathbf{v}) + c_2T(T(\mathbf{v})) &= T(\mathbf{0}) && \text{(def of linear transformation)} \\ c_1T(\mathbf{v}) + c_2T(T(\mathbf{v})) &= \mathbf{0} && \text{(since if } T \text{ is linear, } T(\mathbf{0}) = \mathbf{0}) \\ c_1T(\mathbf{v}) + c_2(\mathbf{0}) &= \mathbf{0} && (T(T\mathbf{v}) = \mathbf{0}) \\ c_1T(\mathbf{v}) &= \mathbf{0} \\ c_1 &= 0 && \text{since } T(\mathbf{v}) \neq 0 \end{aligned}$$

But if $c_1 = 0$, we have

$$0\mathbf{v} + c_2T(\mathbf{v}) = \mathbf{0}$$

so

$$c_2T(\mathbf{v}) = \mathbf{0}$$

which means that $c_2 = 0$, since $T(\mathbf{v}) \neq 0$. Thus we have shown that if

$$c_1\mathbf{v} + c_2T(\mathbf{v}) = \mathbf{0}$$

then $c_1 = c_2 = 0$ and hence $\mathbf{v}$ and $T(\mathbf{v})$ are linearly independent. ⊠