Lenarz | Practice Exam #1 |
Math 210 | September 24, 2007 | Name: ______________

Directions: Answer the following questions on a separate piece of paper. You may use a calculator. If you do not show your work, you will receive no credit. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is much longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Let $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$. Compute $2\mathbf{a} - 3\mathbf{b} + 4\mathbf{c}$.

Solution:

$$\begin{aligned} 2\mathbf{a} - 3\mathbf{b} + 4\mathbf{c} &= 2\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} - \begin{bmatrix} 12 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ -12 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 0 \\ -9 \end{bmatrix} \end{aligned}$$

2. Find a vector $\mathbf{u}$ such that $2\mathbf{u} + 3\mathbf{v} = \mathbf{w}$ where $\mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$.

Solution:

$$\begin{aligned} 2\mathbf{u} + 3\mathbf{v} &= \mathbf{w} \\ 2\mathbf{u} &= \mathbf{w} - 3\mathbf{v} \\ \mathbf{u} &= \frac{1}{2}\mathbf{w} - \frac{3}{2}\mathbf{v} \\ &= \frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{9}{2} \\ -3 \\ \frac{3}{2} \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 5 \\ -\frac{3}{2} \end{bmatrix} \end{aligned}$$

3. Solve for $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$:

$$2\mathbf{x} - \mathbf{a} + 3\mathbf{b} = 2(\mathbf{a} + \mathbf{b}) - 3(\mathbf{x} - \mathbf{b})$$

Solution:

$$\begin{aligned} 2\mathbf{x} - \mathbf{a} + 3\mathbf{b} &= 2(\mathbf{a} + \mathbf{b}) - 3(\mathbf{x} - \mathbf{b}) \\ 2\mathbf{x} - \mathbf{a} + 3\mathbf{b} &= 2\mathbf{a} + 2\mathbf{b} - 3\mathbf{x} + 3\mathbf{b} \\ 5\mathbf{x} &= 3\mathbf{a} + 2\mathbf{b} \\ \mathbf{x} &= \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b} \end{aligned}$$

4. Find all values of the scalar k so that $\mathbf{u} = \begin{bmatrix} k \\ k \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ k-1 \\ 5 \end{bmatrix}$ are orthogonal.

Solution: The two vectors are orthogonal iff their dot product is zero.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 0 \\ -2k + k(k-1) - 10 &= 0 \\ k^2 - 3k - 10 &= 0 \\ (k-5)(k+2) &= 0 \end{aligned}$$

So $k = -2$ and $k = 5$ are the only two values that make $\mathbf{u}$ and $\mathbf{v}$ orthogonal.

5. Suppose I define a new product called the circle dot product so that if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$, then

$$\mathbf{u} \odot \mathbf{v} = 5u_1v_1 + 2u_2v_2$$

Which (if any) of the following statements would be true for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalars c?

(a) $\mathbf{u} \odot \mathbf{v} = \mathbf{v} \odot \mathbf{u}$

Solution:

$$\begin{aligned} \mathbf{u} \odot \mathbf{v} &= 5u_1v_1 + 2u_2v_2 && \text{(definition of } \odot\text{)} \\ &= 5v_1u_1 + 2v_2u_2 && \text{(commutive property of multiplication in } \mathbb{R}\text{)} \\ &= \mathbf{v} \odot \mathbf{u} && \text{(definition of } \odot\text{)} \end{aligned}$$

So $\mathbf{u} \odot \mathbf{v} = \mathbf{v} \odot \mathbf{u}$ is true for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$.

(b) $\mathbf{u} \odot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \odot \mathbf{v} + \mathbf{u} \odot \mathbf{w}$

Solution:

$$\begin{aligned}
\mathbf{u} \odot (\mathbf{v} + \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \odot \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} && \text{(vector addition)} \\
&= 5u_1(v_1 + w_1) + 2u_2(v_2 + w_2) && \text{(definition of } \odot\text{)} \\
&= 5u_1v_1 + 5u_1w_1 + 2u_2v_2 + 2u_2w_2 && \text{(distributive property of multiplication in } \mathbb{R}\text{)} \\
&= (5u_1v_1 + 2u_1v_1) + (5u_1w_1 + 5u_2w_2) && \text{(commutative \& associative properties of addition in } \mathbb{R}\text{)} \\
&= \mathbf{u} \odot \mathbf{v} + \mathbf{u} \odot \mathbf{w} && \text{(definition of } \odot\text{)}
\end{aligned}$$

So $\mathbf{u} \odot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \odot \mathbf{v} + \mathbf{u} \odot \mathbf{w}$ is true for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$.

(c) $(c\mathbf{u}) \odot \mathbf{v} = c(\mathbf{u} \odot \mathbf{v})$

Solution:

$$\begin{aligned}
(c\mathbf{u}) \odot \mathbf{v} &= \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \odot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} && \text{(scalar multiplication)} \\
&= 5(cu_1)v_1 + 2(cu_2)v_2 && \text{(definition of } \odot\text{)} \\
&= c(5u_1v_1) + c(2u_2v_2) && \text{(commutive \& associative properties of multiplication in } \mathbb{R}\text{)} \\
&= c(5u_1v_1 + 2u_2v_2) && \text{(distributive property of multiplication in } \mathbb{R}\text{)} \\
&= c(\mathbf{u} \odot \mathbf{v}) && \text{(definition of } \odot\text{)}
\end{aligned}$$

So $(c\mathbf{u}) \odot \mathbf{v} = c(\mathbf{u} \odot \mathbf{v})$ is true for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$.

(d) $\mathbf{u} \odot \mathbf{u} \geq 0$ and $\mathbf{u} \odot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$

Solution:

$$\begin{aligned}
\mathbf{u} \odot \mathbf{u} &= 5u_1u_1 + 2u_2u_2 && \text{(definition of } \odot\text{)} \\
&= 5u_1^2 + 2u_2^2 \\
&\geq 0 && \text{(sum of squares must be nonnegative)}
\end{aligned}$$

So $\mathbf{u} \odot \mathbf{u} \geq 0$ is true for all $\mathbf{u}$ in $\mathbb{R}^2$.

$$\begin{aligned}
\mathbf{u} \odot \mathbf{u} &= 0 \\
5u_1^2 + 2u_2^2 &= 0
\end{aligned}$$

But the sum of positive terms is zero iff each term is zero. So $5u_1^2 = 0 \Leftrightarrow u_1 = 0$

and $2u_2^2 = 0 \Leftrightarrow u_2 = 0$. Thus $\mathbf{u} = \mathbf{0}$. Hence, $\mathbf{u} \odot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$ is true for all $\mathbf{u}$ in $\mathbb{R}^2$.

6. Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = r \begin{bmatrix} 4 \\ -4 \\ 7 \end{bmatrix}$. Compute the following:

(a) $\mathbf{u} \cdot \mathbf{v}$

Solution:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (1)(4) + (-2)(-4) + (1)(7) \\ &= 4 + 8 + 7 \\ &= 19 \end{aligned}$$

(b) $||\mathbf{u} - \mathbf{v}||$

Solution:

$$\begin{aligned} \mathbf{u} - \mathbf{v} &= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ -4 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 2 \\ -6 \end{bmatrix} \\ ||\mathbf{u} - \mathbf{v}|| &= \sqrt{(-3)^2 + (2)^2 + (-6)^2} \\ &= \sqrt{9 + 4 + 36} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

(c) $\text{proj}_{\mathbf{v}}\mathbf{u}$

Solution:

$$\begin{aligned} ||\mathbf{v}|| &= \sqrt{(4)^2 + (-4)^2 + (7)^2} \\ &= \sqrt{16 + 16 + 49} \\ &= \sqrt{81} \\ &= 9 \\ \text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}\right)\mathbf{v} \\ &= \left(\frac{19}{81}\right)\mathbf{v} \\ &= \begin{bmatrix} \frac{76}{81} \\ -\frac{76}{81} \\ \frac{133}{81} \end{bmatrix} \end{aligned}$$

(d) the cosine of the angle between **u** and **v**

Solution:

$$\begin{aligned} ||\mathbf{u}|| &= \sqrt{(1)^2 + (-2)^2 + (1)^2} \\ &= \sqrt{1 + 4 + 1} \\ &= \sqrt{6} \\ \cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} \\ &= \frac{19}{9\sqrt{6}} \end{aligned}$$

7. Which of the following is **NOT** true for all vectors **u** and **v**:

(a) $||\mathbf{u} - \mathbf{v}|| \leq ||\mathbf{u}|| - ||\mathbf{v}||$

Solution: If we let $\mathbf{u} = \mathbf{0}$ then the statement becomes

$$|| - \mathbf{v}|| \leq -||\mathbf{v}||$$

which is not true because the left side is positive and the right side is negative.

(b) $||\mathbf{u}+\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$

Solution: If we let $\mathbf{u} = \mathbf{0}$ then the statement becomes

$$||\mathbf{v}||^2 = 2||\mathbf{v}||^2$$

which is only true if $||\mathbf{v}||^2 = 0$ (i.e. when $\mathbf{v} = \mathbf{0}$), so it is not true for all vectors.

(c) $\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}-\mathbf{v}||^2$

Solution:

$$\begin{aligned}
\tfrac{1}{4} & \quad ||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}-\mathbf{v}||^2 \\
= & \ \tfrac{1}{4}\Big((\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) - (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\Big) && \text{(def. of mag.)} \\
= & \ \tfrac{1}{4}\Big((\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v})\Big) && \text{(dist. prop.)} \\
= & \ \tfrac{1}{4}\left(4(\mathbf{u}\cdot\mathbf{v})\right) \\
= & \ \mathbf{u}\cdot\mathbf{v}
\end{aligned}$$

So the statement holds for all $\mathbf{u}$ and $\mathbf{v}$.

8. For all vectors $\mathbf{v}$ in $\mathbb{R}^n$, the nonzero vector $\mathbf{u}$ is orthogonal to which of the following?

(a) $\text{proj}_{\mathbf{u}}\mathbf{v}$

Solution:

$$\begin{aligned}
\mathbf{u}\cdot\text{proj}_{\mathbf{u}}\mathbf{v} &= \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u}) \\
&= \mathbf{u}\cdot\mathbf{v}
\end{aligned}$$

So this statement is not true for all $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u}$ is only orthogonal to $\text{proj}_{\mathbf{u}}\mathbf{v}$ under certain conditions (i.e. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal).

(b) $\text{proj}_{\mathbf{v}}\mathbf{u}$

Solution:

$$\begin{aligned} \mathbf{u} \cdot \text{proj}_{\mathbf{v}}\mathbf{u} &= \mathbf{u} \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) (\mathbf{u} \cdot \mathbf{v}) \\ &= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}} \end{aligned}$$

So this statement is not true for all $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u}$ is only orthogonal to $\text{proj}_{\mathbf{v}}\mathbf{u}$ under certain conditions (i.e. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal).

(c) $\mathbf{v} + \text{proj}_{\mathbf{u}}\mathbf{v}$

Solution:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \text{proj}_{\mathbf{u}}\mathbf{v}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \text{proj}_{\mathbf{u}}\mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} \\ &= 2(\mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

So this statement is not true for all $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u}$ is only orthogonal to $\mathbf{v} + \text{proj}_{\mathbf{u}}\mathbf{v}$ under certain conditions (i.e. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal).

(d) $\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}$

Solution:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}) &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \text{proj}_{\mathbf{u}}\mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \\ &= 0 \end{aligned}$$

So this statement is true for all $\mathbf{u}$ and $\mathbf{v}$.

9. Find an equation of the plane through the point $(1, 2, 3)$ which is parallel to the plane $2x - y + z + 2 = 0$ and find the equation of the line through the point $(1, 2, 3)$ that is perpendicular to the plane $2x - y + z + 2 = 0$.

Solution: Parallel planes have parallel normal vectors, so we can use $\mathbf{n} = [2, -1, 1]$

for the normal vector of our plane. Thus we have

$$\begin{aligned} 2(x-1)-(y-2)+(z-3) &= 0 \\ 2x-2-y+2+z-3 &= 0 \\ 2x-y+z &= 3 \end{aligned}$$

The line's direction vector is perpendicular to the plane and hence parallel to the plane's normal vector, so we can take $\mathbf{d} = [2, -1, 1]$. In vector form, the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

10. The line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \\ 7 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$ is perpendicular to a plane containing the point $(4, 6, -1)$. Find an equation of the plane.

Solution: The line's direction vector is perpendicular to the plane and hence parallel to the plane's normal vector, so we can take $\mathbf{n} = [-2, 3, 5]$ for the normal vector of our plane. Thus we have

$$\begin{aligned} -2(x-4)+3(y-6)+5(z-(-1)) &= 0 \\ -2x+8+3y-18+5z+5 &= 0 \\ -2x+3y+5z &= 5 \end{aligned}$$

11. Find all solutions of $3x + 5 = 2$ in $\mathbb{Z}_7$ or show that there are no solutions.

Solution: If we subtract 5 from both sides, we have $2 - 5 = -3 \equiv_7 4$ on the right side. So we are solving the equation

$$3x = 4$$

in $\mathbb{Z}_7$. This becomes a problem of finding a value (in $\mathbb{Z}_7$) for x so that $3x$ has remainder 4 when divided by 7. We can just try all possible values in $\mathbb{Z}_7$ to determine when it would be true. The only value that will work is $x = 6$.

12. Find the check digit that should be appended to the vector $\mathbf{u} = [2\ 5\ 6\ 4\ 5]$ in $\mathbb{Z}_7^5$ if the check vector is $\mathbf{c} = [1\ 1\ 1\ 1\ 1\ 1]$.

Solution:

$$\begin{aligned}[2\ 5\ 6\ 4\ 5\ d] \cdot [1\ 1\ 1\ 1\ 1\ 1] &= 2+5+6+4+5+d \\ &= 1+d \text{ in } \mathbb{Z}_7\end{aligned}$$

So the dot product will be zero if $1+d=0$ in $\mathbb{Z}_7$, which happens when $d=6$.

13. (a) Write down the augmented matrix for the linear system

$$\begin{aligned} x + 2y - 3z &= 1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right]$$

(b) Solve the system using Gaussian elimination or show the system has no solution.

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right] \xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -7 & 11 & 4 \\ 0 & -7 & 11 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -7 & 11 & 4 \\ 0 & 0 & 0 & -7 \end{array}\right]$$

The last row of the augmented matrix gives $0 = -7$, so the system has no solution.

14. (a) Transform the following matrix to reduced row echelon form (showing all steps - not just with a calculator!)

$$\begin{bmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 6 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 6 \\ 1 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 2 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow[R_1 + 3R_3]{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(b) What is the rank of the matrix?

Solution: The rank of the matrix is 3 since there are three nonzero rows of the matrix.

15. Find all solutions of the following systems:

(a)

$$\begin{aligned} 4x + 8y - 12z &= 0 \\ 2x + 2y - 2z &= 0 \\ 5x - 5y + 5z &= 0 \end{aligned}$$

Solution: The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 4 & 8 & -12 & 0 \\ 2 & 2 & -2 & 0 \\ 5 & -5 & 5 & 0 \end{array}\right] \xrightarrow{\text{ref}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

The last line guarantees a unique solution, and since this is a homogeneous system, the only solution is $\mathbf{0}$ (i.e. $x = y = z = 0$)

(b)

$$\begin{array}{rcrcrcr} x & - & 4y & - & z & = & 3 \\ 2x & - & 8y & + & z & = & 9 \\ -x & + & 4y & - & 2z & = & -6 \end{array}$$

Solution: The augmented matrix for the system is

$$\left[\begin{array}{rrr|r} 1 & -4 & -1 & 3 \\ 2 & -8 & 1 & 9 \\ -1 & 4 & -2 & -6 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & -4 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last line guarantees an infinite number of solutions. The matrix gives us the system

$$\begin{array}{rcrcrcr} x & - & 4y & & & = & 4 \\ & & & & z & = & 1 \end{array}$$

Our leading variables are x and z and our free variable is y. So we solve for x and z in terms of y:

$$\begin{array}{rcl} x & = & 4y + 4 \\ z & = & 1 \end{array}$$

If we let $y = t$, we have the solution set $\begin{bmatrix} 4t + 4 \\ t \\ 1 \end{bmatrix}$.

16. Find the line of intersection of the planes $-x + 2y - z + 1 = 0$ and $y + 3z - 1 = 0$.

Solution: This problem is given by a system of linear equations

$$\begin{array}{rcrcrcr} -x & + & 2y & - & z & = & -1 \\ & & y & + & 3z & = & 1 \end{array}$$

which has augmented matrix

$$\left[\begin{array}{rrr|r} -1 & 2 & -1 & -1 \\ 0 & 1 & 3 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 0 & 7 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

The matrix gives us the system

$$\begin{array}{rcrcl} x & & +\ 7z & = & 3 \\ & y & +\ 3z & = & 1 \end{array}$$

Our leading variables are x and y and our free variable is z. So we solve for x and y in terms of z:

$$\begin{aligned} x &= 3 - 7z \\ y &= 1 - 3z \end{aligned}$$

If we let $z = t$, we have the solution set $\begin{bmatrix} 3-7t \\ 1-3t \\ t \end{bmatrix}$.

17. For what values of a and b is the system consistent?

$$\begin{array}{rcrcl} x & - & 3y & = & a \\ -2x & + & 6y & = & b \end{array}$$

Solution: The augmented matrix for this system is

$$\left[\begin{array}{rr|r} 1 & -3 & a \\ -2 & 6 & b \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{rr|r} 1 & -3 & a \\ 0 & 0 & b+2a \end{array}\right]$$

If the last row is all zeros we have an infinite number of solutions and the system is consistent. So if $b + 2a = 0$ we will be consistent, i.e. if $b = -2a$. All other values will have the last row leading to an equation with zero on the left and a nonzero value on the right and hence no solution to the system (inconsistent).

18. For what values of k does the system

$$\begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ 2x & - & y & + & 5z & = & 0 \\ x & - & y & + & kz & = & 0 \end{array}$$

have

(a) no solution

(b) a unique solution

(c) infinitely many solutions

Solution: The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -1 & 5 & 0 \\ 1 & -1 & k & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & k-1 & 0 \end{array}\right]$$

If the last row is all zeros, we will have an infinite number of solutions. So if $k - 1 = 0 \Rightarrow k = 1$ we have an infinite number of solutions.
If the last row has $k - 1 \neq 0 \Rightarrow k \neq 1$, we will have a unique solution.
There are no possible values of k that would lead to no solution (this is a homogeneous system, so it always has at least $\mathbf{0}$ as a solution).

19. Determine if the following sets of vectors are linearly independent. If a set is linearly dependent, find a dependence relationship among the vectors.

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$

Solution: First put the vectors into a matrix as rows:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

Then perform row reduction on A. If we have any free variables (in particular a last row of zeros), we can trace back through the row operations to find a dependence relation. If we have no free variables, the set is linearly independent.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix} \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3''=R_3'-R_2'} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have

$$\begin{aligned} \mathbf{0} &= R_3'' \\ &= R_3' - R_2' \\ &= (R_3 - 3R_1) - (R_2 - 2R_1) \\ &= -R_1 - R_2 + R_3 \\ &= -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix} \end{aligned}$$

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$

Solution: First put the vectors into a matrix as columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{bmatrix}$$

Then perform row reduction on $[A|\mathbf{0}]$. If we have a matrix that corresponds to an infinite number of solutions, we have a dependent set. If we have a unique solution, we have a linearly independent set.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 0 \\ 3 & 7 & 11 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So we have a unique solution, which means our set of vectors is linearly independent.

20. Show that $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$

Solution: We need to show that an arbitrary vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ is a linear combination of the three vectors, i.e. find c_1, c_2 and c_3 so that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This corresponds to a linear system

$$\begin{array}{ccccccc} c_1 & & & & & = & a \\ c_1 & + & c_2 & + & c_3 & = & b \\ c_1 & + & c_2 & - & c_3 & = & c \end{array}$$

Which has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 1 & b \\ 1 & 1 & -1 & c \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b-a \\ 0 & 1 & -1 & c-a \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & -2 & c-b \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 1 & \frac{b-c}{2} \end{array}\right]$$

$$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & \frac{-2a+b+c}{2} \\ 0 & 0 & 1 & \frac{b-c}{2} \end{array}\right]$$

So we have the solution $c_1 = a$, $c_2 = \frac{-2a+b+c}{2}$ and $c_3 = \frac{b-c}{2}$, i.e.

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{-2a+b+c}{2} \right) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{b-c}{2} \right) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

21. Indicate TRUE if true in all cases or FALSE otherwise.

(a) In $\mathbb{R}^n$, if $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, then either $\mathbf{u} = 0$ or $\mathbf{v} = \mathbf{w}$.

Solution: This statement is FALSE. Examining the equation we see that

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathbf{u}\cdot\mathbf{w} \\ \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} &= \mathbf{0} \\ \mathbf{u}\cdot(\mathbf{v} - \mathbf{w}) &= \mathbf{0} \end{aligned}$$

Which means that $\mathbf{u}$ is orthogonal to $\mathbf{v} - \mathbf{w}$. For instance if $\mathbf{u} = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_2$ and $\mathbf{w} = \mathbf{e}_3$ in $\mathbb{R}^n$ where $n \geq 3$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$.

(b) In $\mathbb{R}^n$, $\|\mathbf{v} - \mathbf{w}\| + \|\mathbf{v} + \mathbf{w}\| = \|2\mathbf{v}\|$.

> **Solution:** This statement is FALSE. For instance if $\mathbf{v} = \mathbf{0}$, then the statement says $\| - \mathbf{w}\| + \|\mathbf{w}\| = \mathbf{0}$, which is only true for $\mathbf{w} = \mathbf{0}$.

(c) In $\mathbb{R}^n$, if $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

> **Solution:** This statement is FALSE. $\mathbf{u} \cdot \mathbf{v} = 0$ just means that $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

(d) In $\mathbb{R}^3$, if a line ℓ is parallel to a plane $\mathcal{P}$, then a direction vector $\mathbf{d}$ for ℓ is parallel to a normal vector $\mathbf{n}$ for $\mathcal{P}$.

> **Solution:** This statement is FALSE. It would be true if the line ℓ was *perpendicular* to the plane $\mathcal{P}$.

(e) In $\mathbb{R}^3$, if two planes are not parallel, then they must intersect in a line.

> **Solution:** This statement is TRUE.

(f) If $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $c\|\mathbf{x}\| = \|c\mathbf{x}\|$.

> **Solution:** This statement is FALSE. It would be true if it said $|c|\|\mathbf{x}\| = \|c\mathbf{x}\|$

(g) The rank of a matrix is equal to the number of its nonzero rows.

> **Solution:** This statement is FALSE. It would be true if it said "The rank of a matrix *in row echelon form* is equal to the number of its nonzero rows."

(h) A system of n linear equations in $n + 1$ unknowns always has a solution.

> **Solution:** This statement is FALSE. For example the system
>
> $$\begin{array}{ccccccc} x & + & y & + & z & = & 3 \\ x & + & y & + & z & = & 1 \end{array}$$
>
> has no solution.

Proofs

22. Prove that $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

Solution: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &+ \|\mathbf{u}-\mathbf{v}\|^2 \\
&= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) && \text{(def. of length)}\\
&= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} && \text{(dist. and com. prop.)}\\
&= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 && \text{(def. of length)}
\end{aligned}$$

⊠

23. Prove that if $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then every scalar multiple of $\mathbf{u}$ is also orthogonal to $\mathbf{v}$.

Solution: Suppose $\mathbf{u}$ is orthogonal to $\mathbf{v}$. Then $\mathbf{u}\cdot\mathbf{v} = 0$ by the definition of orthogonal. So we have

$$\begin{aligned}
(c\mathbf{u})\cdot\mathbf{v} &= c(\mathbf{u}\cdot\mathbf{v}) && \text{(property of dot product)}\\
&= c(0) && \text{(since } \mathbf{u}\cdot\mathbf{v} = 0)\\
&= 0
\end{aligned}$$

So by the definition of orthogonality, $c\mathbf{u}$ is orthogonal to $\mathbf{v}$. ⊠

24. Prove that $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}\mathbf{v}) = \text{proj}_{\mathbf{u}}\mathbf{v}$.

Solution:

$$\begin{aligned}
\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}(\mathbf{v})) &= \text{proj}_{\mathbf{u}}\left(\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right) && \text{(def. of proj.)}\\
&= \left(\frac{\mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} && \text{(def. of proj.)}\\
&= \left(\frac{\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u})}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} && \text{(prop. of dot prod.)}\\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\\
&= \text{proj}_{\mathbf{u}}\mathbf{v} && \text{(def. of proj.)}
\end{aligned}$$

⊠

25. Prove that a set of vectors containing two equal vectors is linearly dependent.

Solution: Suppose S is a set of vectors containing two equal vectors. For instance

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}, \mathbf{u}\}$$

Then if we let $c_1 = c_2 = \ldots = c_k = 0$ and $c_{k+1} = -c_{k+2}$ where $c_{k+2} \neq 0$, we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k + c_{k+1}\mathbf{u} + c_{k+2}\mathbf{u} = -c_{k+2}\mathbf{u} + c_{k+2}\mathbf{u} = \mathbf{0}$$

which shows that the set S is linearly dependent. ⊠

26. Prove that if $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_k$ are linearly independent vectors, then $\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, \ldots, c_k\mathbf{u}_k\}$ is a linearly independent set, where c_i are nonzero scalars.

Solution: Suppose $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_k$ are linearly independent vectors. Let b_i and c_i be sets of scalars with $c_i \neq 0$ for $i = 1, \ldots, k$. Then

$$b_1(c_1\mathbf{u}_1) + b_2(c_2\mathbf{u}_2) + \ldots + b_k(c_k\mathbf{u}_k) = \mathbf{0}$$

iff

$$(b_1c_1)\mathbf{u}_1 + (b_2c_2)\mathbf{u}_2 + \ldots + (b_kc_k)\mathbf{u}_k = \mathbf{0}$$

Since $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_k$ are linearly independent vectors, $b_ic_i = 0$ for $i = 1, \ldots, k$. But $c_i \neq 0$ for $i = 1, \ldots, k$, so $b_i = 0$ for $i = 1, \ldots, k$. Thus $c_1\mathbf{u}_1$, $c_2\mathbf{u}_2$, …, $c_k\mathbf{u}_k$ are linearly independent vectors. ⊠

27. Prove the following by induction

(a) $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for $n \geq 1$

Solution:
Base step: $n = 1$

$$1 = 1^2 \checkmark$$

Induction Step: Assume that

$$1 + 3 + 5 + \ldots + (2n-1) = n^2 \text{ (Induction Hypothesis)}$$

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2n-1) + [2(n+1)-1] &= n^2 + [2(n+1)-1] \quad \text{(by Ind. Hyp.)} \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

⊠

(b) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for $n \geq 1$

Solution:
Base step: $n = 1$

$$\begin{aligned} \frac{(1)(2(1)-1)(2(1)+1)}{3} &= \frac{(1)(1)(3)}{3} \\ &= 1 \\ &= 1^2 \checkmark \end{aligned}$$

Induction Step: Assume that

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \text{ (Induction Hypothesis)}$$

$$\begin{aligned} 1^2 + 3^2 + \quad 5^2 &\quad + \ldots + (2n-1)^2 + [2(n+1)-1]^2 \\ &= \frac{n(2n-1)(2n+1)}{3} + [2(n+1)-1]^2 \quad \text{(by Ind. Hyp.)} \\ &= \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2 \\ &= (2n+1)\left(\frac{n(2n-1)}{3} + (2n+1)\right) \\ &= (2n+1)\left(\frac{n(2n-1)+3(2n+1)}{3}\right) \\ &= (2n+1)\left(\frac{2n^2+5n+3}{3}\right) \\ &= (2n+1)\left(\frac{(2n+3)(n+1)}{3}\right) \\ &= \frac{(n+1)[2(n+1)-1][2(n+1)+1]}{3} \qquad \boxtimes \end{aligned}$$

(c) $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{n(n+1)} = \dfrac{n}{n+1}$ for $n \geq 1$

Solution:
Base step: $n = 1$

$$\begin{aligned} \frac{(1)}{(1)+1} &= \frac{1}{2} \\ &= \frac{1}{1(2)} \checkmark \end{aligned}$$

Induction Step: Assume that

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \text{ (Induction Hypothesis)}$$

$$\begin{aligned}
\frac{1}{1(2)} + \frac{1}{2(3)} &+ \frac{1}{3(4)} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)[(n+1)+1]} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)[(n+1)+1]} \quad \text{(by Ind. Hyp.)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{1}{n+1}\left(n + \frac{1}{n+2}\right) \\
&= \frac{1}{n+1}\left(\frac{n(n+2)+1}{n+2}\right) \\
&= \frac{1}{n+1}\left(\frac{n^2+2n+1}{n+2}\right) \\
&= \frac{1}{n+1}\left(\frac{(n+1)^2}{n+2}\right) \\
&= \frac{n+1}{n+2} \\
&= \frac{n+1}{(n+1)+1}
\end{aligned}$$

⊠