Lenarz | Exam #1 |
Math 210 | October 1, 2007 | Name: ____________________

Directions: Answer the following questions in the space provided. You may use a calculator. If you do not show your work, you will receive no credit. The point value of each question is indicated. **WRITE YOUR FINAL ANSWER ON THE ANSWER LINE WHEN APPROPRIATE. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!**

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1. If $\mathbf{u} = [2, 3, -1]$ and $\mathbf{v} = [-1, 5, 0]$, find

   (a) (5 points) $\mathbf{u} \cdot \mathbf{v}$

   Solution:

   $$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (2)(-1) + (3)(5) + (-1)(0) \\ &= -2 + 15 + 0 \\ &= 13 \end{aligned}$$

   (b) (5 points) $\|2\mathbf{u} - \mathbf{v}\|$

   Solution:

   $$\begin{aligned} \|2\mathbf{u} - \mathbf{v}\| &= \|[4, 6, -2] - [-1, 5, 0]\| \\ &= \|[5, 1, -2]\| \\ &= \sqrt{5^2 + 1^2 + (-2)^2} \\ &= \sqrt{25 + 1 + 4} \\ &= \sqrt{30} \end{aligned}$$

   (c) (5 points) $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$

   Solution:

   $$\begin{aligned} \mathrm{proj}_{\mathbf{v}}\mathbf{u} &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \\ &= \frac{13}{26}\mathbf{v} \\ &= \frac{1}{2}\mathbf{v} \\ &= \left[-\frac{1}{2}, \frac{5}{2}, 0\right] \end{aligned}$$

(d) (5 points) cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

Solution:

$$\begin{aligned} \cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\ &= \frac{13}{\sqrt{14}\sqrt{26}} \\ &= \frac{13}{2\sqrt{91}} \end{aligned}$$

2. (9 points) For what value of a are $[a,-2,3]$ and $[a,5,a]$ orthogonal?

Solution: The two vectors are orthogonal iff their dot product is zero:

$$\begin{aligned} 0 &= [a,-2,3]\cdot[a,5,a] \\ &= (a)(a)+(-2)(5)+(3)(a) \\ &= a^2+3a-10 \\ &= (a+5)(a-2) \end{aligned}$$

So the two vectors are orthogonal when $a=-5$ or $a=2$.

3. (15 points) Consider the linear system

$$\begin{array}{ccccccc} x & & & + & kz & = & 1 \\ kx & + & y & + & (k^2-7)z & = & k-1 \\ 2x & + & ky & + & (k^2-14)z & = & 0 \end{array}$$

(a) For what value(s) of k does the system have a unique solution?

(b) For what value(s) of k does the system have no solution?

(c) For what value(s) of k does the system have infinitely many solutions?

Solution: First we find a row echelon form of the associated augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & k & 1 \\ k & 1 & k^2-7 & k-1 \\ 2 & k & k^2-14 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-kR_1} \left[\begin{array}{ccc|c} 1 & 0 & k & 1 \\ 0 & 1 & -7 & -1 \\ 0 & k & k^2-2k-14 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-kR_2} \left[\begin{array}{ccc|c} 1 & 0 & k & 1 \\ 0 & 1 & -7 & -1 \\ 0 & k & k^2+5k-14 & k-2 \end{array}\right]$$

(Notice that these row operations are valid only when $k \neq 0$, but when $k = 0$, the rref of the matrix shows a unique solution.) There will be an infinite number of solutions when the last row is all zeros; that is when $k^2 + 5k - 14 = (k+7)(k-2) = 0$ and $k - 2 = 0$. This happens only when $k = 2$.
There will be no solution if $k^2 + 5k - 14 = (k+7)(k-2) = 0$ but $k - 2 \neq 0$. This happens when $k = -7$.
The only other option is to have a unique solution, which will occur when $k \neq -7, 2$.

4. Are the following vectors linearly dependent? If they are, find a dependence relationship.

(a) (10 points) $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 4 \\ -3 \end{bmatrix}$

Solution: We need to find scalars c_1, c_2, c_3, and c_4 so that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 6 \\ -5 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So the corresponding linear system is

$$\begin{array}{rcrcrcrcl} c_1 & - & c_2 & + & c_3 & + & 6c_4 & = & 0 \\ -c_1 & + & c_2 & & & - & 5c_4 & = & 0 \\ c_1 & & & + & c_3 & + & 4c_4 & = & 0 \\ & & c_2 & - & c_3 & - & 3c_4 & = & 0 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 6 & 0 \\ -1 & 1 & 0 & -5 & 0 \\ 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & -3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which corresponds to the system

$$\begin{array}{rcrcl} c_1 & & & + & 3c_4 & = & 0 \\ & c_2 & & - & 2c_4 & = & 0 \\ & & c_3 & + & c_4 & = & 0 \end{array}$$

So the solution is

$$\begin{bmatrix} -3t \\ 2t \\ -t \\ t \end{bmatrix}$$

telling us that the vectors are linearly dependent have dependence relationship

$$-3\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}+2\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}-\begin{bmatrix}1\\0\\1\\-1\end{bmatrix}+\begin{bmatrix}6\\-5\\4\\-3\end{bmatrix}=\mathbf{0}$$

or

$$3\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}-2\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}+\begin{bmatrix}1\\0\\1\\-1\end{bmatrix}=\begin{bmatrix}6\\-5\\4\\-3\end{bmatrix}$$

(b) (10 points) $\begin{bmatrix}1\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$

Solution: We need to find scalars c_1, c_2, c_3, and c_4 so that

$$c_1\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}-1\\1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}1\\0\\1\\-1\end{bmatrix}+c_4\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

So the corresponding linear system is

$$\begin{array}{rcrcrcrcl} c_1 & - & c_2 & + & c_3 & & & = & 0\\ -c_1 & + & c_2 & & & + & c_4 & = & 0\\ c_1 & & & + & c_3 & - & c_4 & = & 0\\ & & c_2 & - & c_3 & + & c_4 & = & 0 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0\\ -1 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Implying that the solution is $c_1 = c_2 = c_3 = c_4 = 0$. So the set of vectors is linearly independent.

5. (10 points) Indicate TRUE if true in all cases or FALSE otherwise. **Justify your answers**. (Two points each.)

(a) In $\mathbb{R}^n$, if $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{w}$.

> **Solution:** This statement is FALSE. For example, if $\mathbf{u} = [1, 0, 0]$, $\mathbf{v} = [0, 1, 0]$ and $\mathbf{w} = [0, 0, 1]$, then $\mathbf{u} \cdot \mathbf{v} = 0 = \mathbf{u} \cdot \mathbf{w}$, but we have $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{w}$.

(b) In $\mathbb{R}^n$, $\|\mathbf{v} - \mathbf{w}\| + \|\mathbf{v} + \mathbf{w}\| = \|2\mathbf{v}\|$.

> **Solution:** This statement is FALSE. If we let $\mathbf{v} = [1, 0, 0]$ and $\mathbf{w} = [0, 1, 0]$, then we have
>
> $$\begin{aligned} \|\mathbf{v} - \mathbf{w}\| + \|\mathbf{v} + \mathbf{w}\| &= \|[1, -1, 0]\| + \|[1, 1, 0]\| \\ &= 2\sqrt{2} \end{aligned}$$
>
> But $\|2\mathbf{v}\| = \|[2, 0, 0]\| = 2$, so the two quantities are not equal.

(c) In $\mathbb{R}^3$, if a line ℓ is parallel to a plane $\mathcal{P}$, then a direction vector $\mathbf{d}$ for ℓ is perpendicular to a normal vector $\mathbf{n}$ for $\mathcal{P}$.

> **Solution:** This statement is TRUE. If ℓ is parallel to P, then $\mathbf{d}$ is parallel to all vectors in P and hence orthogonal to $\mathbf{n}$ by definition.

(d) If $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $c\|\mathbf{x}\| = \|c\mathbf{x}\|$.

> **Solution:** This statement is FALSE. The correct statement would be: "If $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $|c|\|\mathbf{x}\| = \|c\mathbf{x}\|$."

(e) The rank of a matrix is equal to the number of nonzero rows in its reduced row echelon form.

> **Solution:** This statement is TRUE. This is our current definition of rank.

6. (13 points) Use induction to prove that $1+3+5+\cdots+(2n-1)=n^2$ for $n \geq 1$.

Solution:
Base step: $n=1$

$$1 = 1^2 \checkmark$$

Induction Step: Assume that

$$1+3+5+\cdots+(2k-1)=k^2 \text{ (Induction Hypothesis)}$$

$$\begin{aligned}
1+3+5+\cdots &+(2k-1)+\Big(2(k+1)-1\Big) \\
&= \Big(1+3+5+\cdots+(2k-1)\Big)+(2k+1) && \text{(assoc. prop.)} \\
&= k^2+(2k+1) && \text{(Ind. Hyp.)} \\
&= k^2+2k+1 \\
&= (k+1)^2
\end{aligned}$$

⊠

7. (13 points) Choose only **ONE** of the following statements to prove.
   (a) Prove that $\text{proj}_{\mathbf{u}}(\mathbf{v}-\text{proj}_{\mathbf{u}}\mathbf{v})=\mathbf{0}$.

Solution:

$$\begin{aligned}
\text{proj}_{\mathbf{u}}(\mathbf{v}-\text{proj}_{\mathbf{u}}(\mathbf{v})) &= \text{proj}_{\mathbf{u}}\left(\mathbf{v}-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right) && \text{(def. of proj.)} \\
&= \left(\frac{\mathbf{u}\cdot\left(\mathbf{v}-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right)}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} && \text{(def. of proj.)} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} && \text{(distributive prop.)} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u})}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} && \text{(prop. of dot prod.)} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\
&= (0)\mathbf{u} \\
&= \mathbf{0}
\end{aligned}$$

⊠

(b) Prove that a set of vectors containing two equal vectors is linearly dependent.

Solution: Suppose S is a set of vectors containing two equal vectors. For instance

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}, \mathbf{u}\}$$

Then if we let $c_1 = c_2 = \ldots = c_k = 0$ and $c_{k+1} = -c_{k+2}$ where $c_{k+2} \neq 0$, we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k + c_{k+1}\mathbf{u} + c_{k+2}\mathbf{u} = -c_{k+2}\mathbf{u} + c_{k+2}\mathbf{u} = \mathbf{0}$$

which shows that the set S is linearly dependent. ⊠

(c) Prove that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent vectors, then $\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, \ldots, c_k\mathbf{u}_k\}$ is a linearly independent set, where c_i are nonzero scalars.

Solution: Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent vectors. Let b_i and c_i be sets of scalars with $c_i \neq 0$ for $i = 1, \ldots, k$. Then

$$b_1(c_1\mathbf{u}_1) + b_2(c_2\mathbf{u}_2) + \ldots + b_k(c_k\mathbf{u}_k) = \mathbf{0}$$

iff

$$(b_1c_1)\mathbf{u}_1 + (b_2c_2)\mathbf{u}_2 + \ldots + (b_kc_k)\mathbf{u}_k = \mathbf{0}$$

Since $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent vectors, $b_ic_i = 0$ for $i = 1, \ldots, k$. But $c_i \neq 0$ for $i = 1, \ldots, k$, so $b_i = 0$ for $i = 1, \ldots, k$. Thus $c_1\mathbf{u}_1$, $c_2\mathbf{u}_2$, $\ldots$, $c_k\mathbf{u}_k$ are linearly independent vectors. ⊠

8. (7 BONUS points) A book publisher publishes a potential best seller in three different bindings: paperback, book club, and deluxe. Each paperback book requires 1 minute for sewing and 2 minutes for gluing. Each book club book requires 2 minutes for sewing and 4 minutes for gluing. Each deluxe book requires 3 minutes for sewing and 5 minutes for gluing. If the sewing plant is available 6 hours per day and the gluing plant is available 11 hours per day, how many books of each type can be produced per day so that the plants are fully utilized and the number of paperbacks is four times the number of book club books?

Solution: Let p be the number of paperbacks, b the number of book club books, and d the number of deluxe books. Then we have the following system

$$\begin{array}{rcrcrcr} p & + & 2b & + & 3d & = & 360 \\ 2p & + & 4b & + & 5d & = & 660 \\ p & - & 4b & & & = & 0 \end{array}$$

Which has augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 360 \\ 2 & 4 & 5 & 660 \\ 1 & -4 & 0 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

So we have $p = 120$, $b = 30$ and $d = 60$.