Lenarz - Math 122

# §9.4 - Exponential Growth & Decay

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The mathematical model

$$\frac{dy}{dx} = ky$$

where $k$ is a constant of proportionality, models a wide range of naturally occurring phenomena: population growth (under ideal circumstances), radioactive decay, heating and cooling, continuously compounded interest, etc.

The natural growth equation is found by solving this differential equation:

$$\begin{aligned}
\frac{dy}{dx} &= ky \\
\frac{1}{y}\ dy &= k\ dx \\
\int \frac{1}{y}\ dy &= \int k\ dx \\
\ln|y| &= kx + C \\
|y| &= e^{kx+C} \\
y &= e^{kx}e^{C} \\
y &= Ce^{kx}
\end{aligned}$$

Let $y_0 = y(0)$ be the initial condition. Then we have

$$\begin{aligned}
y_0 &= y(0) \\
y_0 &= Ce^{k(0)} \\
y_0 &= C(1) \\
y_0 &= C
\end{aligned}$$

So the natural growth equation is

$$y(t) = y_0 e^{kx}$$

where $y_0$ is the initial condition and $k$ is a constant.

## Population Growth

If $P(t)$ is the number of individuals in a population having constant birth rate $\beta$ and death rate $\delta$ (in births or deaths per individual per unit of time), then during a short time interval $\Delta t$, there are approximately $\beta P(t)\Delta t$ births and $\delta P(t)\Delta t$ deaths. So the change in the population is approximately

$$\Delta P(t) \approx \beta P(t)\Delta t - \delta P(t)\Delta t = (\beta - \delta)P(t)\Delta t$$

which gives us the average rate of change

$$\frac{\Delta P}{\Delta t} = (\beta - \delta)P(t)$$

Therefore the instantaneous rate of change is

$$\frac{dP}{dt} = \lim_{t \to 0} \frac{\Delta P}{\Delta t} = (\beta - \delta)P = kP$$

where $k = \beta - \delta$. $k$ is referred to as the relative growth rate. So the growth equation is

$$P(t) = P_0 e^{kt}$$

where $P_0$ is the initial population and $k$ is the relative growth rate.

**Example 1:** A bacteria culture grows with constant relative growth rate. After 2 hours, there are 600 bacteria and after 8 hours the count is 75,000.

(a) Find the initial population.

(b) Find an expression for the population after $t$ hours.

(c) Find the number of cells after 5 hours.

(d) Find the rate of growth after 5 hours.

(e) When will the population reach 200,000?

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## Radioactive Decay

Let $m(t)$ be the mass of a radioactive element remaining from an initial mass $m_0$ at time $t$. It has been determined experimentally that a constant proportion of the mass will decay (into another isotope of the element) during each unit of time. In other words, radioactive substances decay at a rate proportional to the remaining mass. Thus we have

$$\frac{dm}{dt} = km$$

where $k$ is a constant of proportionality (referred to as the relative rate of decay). Physicists express the rate of decay in terms of a half-life, that is the time required for half of a sample to decay. So the equation for radioactive decay is

$$m(t) = m_0 e^{kt}$$

where $m_0$ is the initial mass of the substance and $k$ is a constant (determined using half-life).

**Example 2:** We know that the half-life of carbon-14 is 5,730 years. The Shroud of Turin is an ancient artifact that many believe to be the burial shroud of Jesus Christ. In 1988, the Vatican consented to give a few fibers to scientists to carbon-date. They found that the fibers had 92% of the original $^{14}C$ left. Using our model of radioactive decay, what was the approximate age of the shroud in 1988? (To complicate matters, microbiologists have discovered remains of centuries-old fungus in the fibers. This fungus could have contaminated the $^{14}C$ sample. If this turned out to be the case, then the age of the shroud would still be a mystery.)

## Newton's Law of Cooling

Newton's Law of Cooling (and Heating) states that the rate of change of the temperature of an object is proportional to the temperature difference between the object and its surroundings, provided this difference is not too large. If we let $T(t)$ be the temperature of the object at time $t$ and $E$ be the temperature of the environment, then Newton's Law is given by

$$\frac{dT}{dt} = k(T - E)$$

where $k$ is a constant of proportionality.

## Continuously Compounded Interest

If $A_0$ is the amount (in dollars) invested at time $t$ (in years) at an interest rate $r$, compounded $n$ times a year, then after $t$ years the investment is worth

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

If we let the number of compounding periods increase without bound (called compounding continuously), then we have

$$\begin{aligned} A(t) &= \lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{n\to\infty} A_0 \left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0 \left[\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0 \left[\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \\ &= A_0 e^{rt} \end{aligned}$$

where $m = \dfrac{n}{r}$. If we differentiate this equation we arrive at

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

so it is also modeled by the natural growth model.