

1. $f(x) = 10 + 27x - x^3$, $[0, 4]$

$$f'(x) = 27 - 3x^2$$

To find the critical points we look for places where f' is undefined or where $f' = 0$. f' is always defined, so we need only find the places where $f' = 0$.

$$\begin{aligned} 27 - 3x^2 &= 0 \\ 27 &= 3x^2 \\ 9 &= x^2 \\ \pm 3 &= x \end{aligned}$$

$x = 3$ is the only critical point in the interval $[0, 4]$. To find local extrema, use the second derivative test for critical points in the interval:

$$f''(x) = -6x$$

and

$$f''(3) = -18 < 0$$

Therefore $f(3) = 64$ is a local max. To find absolute extrema, we look at the values of the function at the endpoints and critical points

$$\begin{aligned} f(0) &= 10 + 27(0) - (0)^3 \\ &= 10 \\ f(3) &= 10 + 27(3) - (3)^3 \\ &= 64 \\ f(4) &= 10 + 27(4) - (4)^3 \\ &= 54 \end{aligned}$$

So the absolute max value is 64 and occurs at $x = 3$. The absolute min value is 10 and occurs at $x = 0$.

2. $f(x) = x - \sqrt{x}$, $[0, 4]$

$$f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}}$$

To find the critical points we look for places where f' is undefined or where $f' = 0$. f' is not defined when $2\sqrt{x} = 0$ which is when $x = 0$. The places where $f' = 0$ are

$$\begin{aligned} 2\sqrt{x} - 1 &= 0 \\ 2\sqrt{x} &= 1 \\ \sqrt{x} &= \frac{1}{2} \\ x &= \frac{1}{4} \end{aligned}$$

$x = \frac{1}{4}$ is the only critical points in the interval $[0, 4]$. To find local extrema, use the second derivative test for critical points in the interval:

$$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} = \frac{1}{4x^{\frac{3}{2}}}$$

and

$$f''\left(\frac{1}{4}\right) = 2 > 0$$

Therefore $f\left(\frac{1}{4}\right) = -\frac{1}{4}$ is a local min. To find absolute extrema, we look at the values of the function at the endpoints and critical points

$$\begin{aligned} f(0) &= 0 - \sqrt{0} \\ &= 0 \\ f\left(\frac{1}{4}\right) &= \frac{1}{4} - \sqrt{\frac{1}{4}} \\ &= -\frac{1}{4} \\ f(4) &= 4 - \sqrt{4} \\ &= 2 \end{aligned}$$

So the absolute max value is 2 and occurs at $x = 4$. The absolute min value is $-\frac{1}{4}$ and occurs at $x = \frac{1}{4}$.

3. $f(x) = \frac{x}{x^2 + x + 1}, [-2, 0]$

$$f'(x) = \frac{(x^2 + x + 1)(1) - x(2x + 1)}{(x^2 + x + 1)^2} = \frac{-(x^2 - 1)}{(x^2 + x + 1)^2}$$

To find the critical points we look for places where f' is undefined or where $f' = 0$. f' is not defined when $x^2 + x + 1 = 0$ which never happens. The places where $f' = 0$ are

$$\begin{aligned} -(x^2 - 1) &= 0 \\ -(x - 1)(x + 1) &= 0 \end{aligned}$$

$x = -1$ is the only critical point in the interval $[-2, 0]$. To find local extrema, use the

second derivative test for critical points in the interval:

$$\begin{aligned}
 f''(x) &= \frac{(x^2 + x + 1)^2(-2x) - (-x^2 + 1)(2(x^2 + x + 1)(2x + 1))}{[(x^2 + x + 1)^2]^2} \\
 &= \frac{(x^2 + x + 1)[(x^2 + x + 1)(-2x) - 2(2x + 1)(-x^2 + 1)]}{(x^2 + x + 1)^4} \\
 &= \frac{-2x^3 - 2x^2 - 2x + 4x^3 - 4x + 2x^2 - 2}{(x^2 + x + 1)^3} \\
 &= \frac{2x^3 - 6x - 2}{(x^2 + x + 1)^3} \\
 &= \frac{2(x^3 - 3x - 1)}{(x^2 + x + 1)^3}
 \end{aligned}$$

and

$$f''(-1) = 2 > 0$$

Therefore $f(-1) = -1$ is a local min. To find absolute extrema, we look at the values of the function at the endpoints and critical points

$$\begin{aligned}
 f(-2) &= \frac{-2}{(-2)^2 + (-2) + 1} \\
 &= -\frac{2}{3} \\
 f(-1) &= \frac{-1}{(-1)^2 + (-1) + 1} \\
 &= -1 \\
 f(0) &= \frac{0}{(0)^2 + (0) + 1} \\
 &= 0
 \end{aligned}$$

So the absolute max value is 0 and occurs at $x = 0$. The absolute min value is -1 and occurs at $x = -1$.

4. $f(x) = (x^2 + 2x)^3, [-2, 1]$

$$f'(x) = 3(x^2 + 2x)^2(2x + 2)$$

To find the critical points we look for places where f' is undefined or where $f' = 0$. f' is always defined so we look for the places where $f' = 0$.

$$\begin{aligned}
 3(x^2 + 2x)^2(2x + 2) &= 0 \\
 3[x(x + 2)]^2(2)(x + 1) &= 0 \\
 6x^2(x + 2)^2(x + 1) &= 0
 \end{aligned}$$

$x = 0$ and $x = -1$ are the only critical points in the interval $[-2, 1]$. To find local extrema, use the first derivative test for critical points in the interval:

interval	$(-2, -1)$	$(-1, 0)$	$(0, 1)$
f'	-	+	+
I/D	D	I	I

Therefore $f(-1) = -1$ is a local min and $f(0) = 0$ is not an extremum. To find absolute extrema, we look at the values of the function at the endpoints and critical points

$$\begin{aligned}
 f(-2) &= ((-2)^2 + 2(-2))^3 \\
 &= 0 \\
 f(-1) &= ((-1)^2 + 2(-1))^3 \\
 &= -1 \\
 f(0) &= ((0)^2 + 2(0))^3 \\
 &= 0 \\
 f(1) &= ((1)^2 + 2(1))^3 \\
 &= 27
 \end{aligned}$$

So the absolute max value is 27 and occurs at $x = 1$. The absolute min value is -1 and occurs at $x = -1$.

5. $f(x) = x + \sin 2x$, $[0, \pi]$

$$f'(x) = 1 + 2 \sin 2x \cos 2x = 1 + \sin 4x$$

To find the critical points we look for places where f' is undefined or where $f' = 0$. f' is always defined so we look for the places where $f' = 0$.

$$\begin{aligned}
 1 + \sin 4x &= 0 \\
 \sin 4x &= -1 \\
 4x &= \frac{3\pi}{2} + 2n\pi \\
 x &= \frac{3\pi}{8} + \frac{n\pi}{2}
 \end{aligned}$$

where n is any integer. $x = \frac{3\pi}{8}$ is the only critical point in the interval $[0, \pi]$. To find local extrema, use the first derivative test for critical points in the interval:

interval	$(0, \frac{3\pi}{8})$	$(\frac{3\pi}{8}, \pi)$
f'	+	+
I/D	I	I

Therefore $f(\frac{3\pi}{8}) = \frac{3\pi}{8} + \frac{\sqrt{2}}{2}$ is not an extremum. To find absolute extrema, we look at

the values of the function at the endpoints and critical points

$$\begin{aligned}
 f(0) &= (0) + \sin 2(0) \\
 &= 0 \\
 f\left(\frac{3\pi}{8}\right) &= \left(\frac{3\pi}{8}\right) + \sin 2\left(\frac{3\pi}{8}\right) \\
 &= \frac{3\pi}{8} + \frac{\sqrt{2}}{2} \\
 f(\pi) &= (\pi) + \sin 2(\pi) \\
 &= \pi
 \end{aligned}$$

So the absolute max value is π and occurs at $x = \pi$. The absolute min value is 0 and occurs at $x = 0$.

6. $f(x) = \frac{\ln x}{x^2}, [1, 3]$

$$f'(x) = \frac{(x^2)\left(\frac{1}{x}\right) - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3}$$

To find the critical points we look for places where f' is undefined or where $f' = 0$. f' is not defined when $x^3 = 0$ which happens when $x = 0$, but $x = 0$ is not in the domain of f so we don't need to consider it. The places where $f' = 0$ are

$$\begin{aligned}
 1 - 2 \ln x &= 0 \\
 1 &= 2 \ln x \\
 \frac{1}{2} &= \ln x \\
 e^{\frac{1}{2}} &= e^{\ln x} \\
 e^{\frac{1}{2}} &= x
 \end{aligned}$$

$x = e^{\frac{1}{2}}$ is the only critical point in the interval $[1, 3]$. To find local extrema, use the second derivative test for critical points in the interval:

$$\begin{aligned}
 f''(x) &= \frac{x^3\left(-\frac{2}{x}\right) - (1 - 2 \ln x)(3x^2)}{(x^3)^2} \\
 &= \frac{-2x^2 - 3x^2(1 - 2 \ln x)}{x^6} \\
 &= \frac{-x^2(2 + 3(1 - 2 \ln x))}{x^6} \\
 &= -\frac{5 - 6 \ln x}{x^4}
 \end{aligned}$$

and

$$f''(e^{\frac{1}{2}}) = \frac{2}{e^2} > 0$$

Therefore $f(e^{\frac{1}{2}}) = \frac{1}{2e}$ is a local min. To find absolute extrema, we look at the values of the function at the endpoints and critical points

$$\begin{aligned} f(1) &= \frac{\ln(1)}{(1)^2} \\ &= 0 \\ f(e^{\frac{1}{2}}) &= \frac{\ln(e^{\frac{1}{2}})}{(e^{\frac{1}{2}})^2} \\ &= \frac{1}{2e} \\ f(3) &= \frac{\ln(3)}{(3)^2} \\ &= \frac{\ln 3}{9} \end{aligned}$$

So the absolute max value is $\frac{1}{2e}$ and occurs at $x = e^{\frac{1}{2}}$. The absolute min value is 0 and occurs at $x = 1$.

7. skip

8. skip

9. skip

10. skip

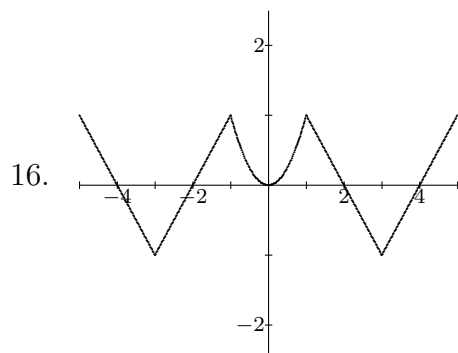
11. skip

12. skip

13. skip

14. skip

15. Look in the back of your book



17. Look in the back of your book

18. (a) increasing on $(-2, 0)$ and $(4, \infty)$; decreasing on $(-\infty, -2)$ and $(0, 4)$
 (b) Local max at $x = 0$, Local min at $x = -2$ and $x = 4$.
 (c) See me
 (d) See me

19. skip

20. $y = x^3 - 6x^2 - 15x + 4$

- (a) Domain: all real numbers
 (b) Intercepts: y -intercept at $y = 0^3 - 6(0)^2 - 15(0) + 4 = 4$ so at the point $(0, 4)$.
 x -intercepts (found using calculator) are approximately at $x = -2.09$, $x = 0.24$ and $x = 7.85$
 (c) Asymptotes: No vertical asymptotes.
 Horizontal Asymptotes:

$$\lim_{x \rightarrow \infty} (x^3 - 6x^2 - 15x + 4) = \infty$$

$$\lim_{x \rightarrow -\infty} (x^3 - 6x^2 - 15x + 4) = -\infty$$

So no horizontal asymptotes.

- (d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

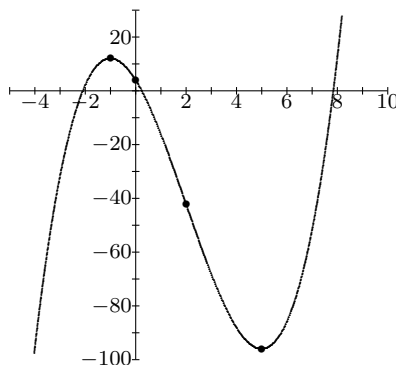
$$\begin{aligned} y' &= 3x^2 - 12x - 15 \\ &= 3(x^2 - 4x - 5) \\ &= 3(x - 5)(x + 1) \\ y'' &= 6x - 12 \\ &= 6(x - 2) \end{aligned}$$

y' is defined everywhere and $y' = 0$ when $x = -1, 5$. y'' is defined everywhere and $y'' = 0$ when $x = 2$. So we have the following:

interval	$(-\infty, -1)$	$(-1, 2)$	$(2, 5)$	$(5, \infty)$
y'	+	-	-	+
y''	-	-	+	+
I/D	I	D	D	I
CU/CD	CD	CD	CU	CU

So we have a local max at $(-1, 12)$, a local min at $(5, -96)$ and an inflection point at $(2, -42)$.

(e) So our graph looks like



21. skip

$$22. y = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$$

(a) Domain: $x \neq \pm 1$

(b) Intercepts: y -intercept at $y = \frac{1}{1-(0)^2} = 1$ so at the point $(0, 1)$.

x -intercepts when $0 = \frac{1}{1-x^2}$, which never happens. So no x -intercepts.

(c) Asymptotes: Vertical asymptotes: $x = \pm 1$

$$\lim_{x \rightarrow -1^-} \frac{1}{(1-x)(1+x)} = -\infty$$

$$\lim_{x \rightarrow -1^+} \frac{1}{(1-x)(1+x)} = +\infty$$

$$\lim_{x \rightarrow 1^-} \frac{1}{(1-x)(1+x)} = +\infty$$

$$\lim_{x \rightarrow 1^+} \frac{1}{(1-x)(1+x)} = -\infty$$

Horizontal Asymptotes:

$$\lim_{x \rightarrow \infty} \frac{1}{1-x^2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2} - 1} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{1-x^2} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2} - 1} = 0$$

So horizontal asymptote of $y = 0$.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

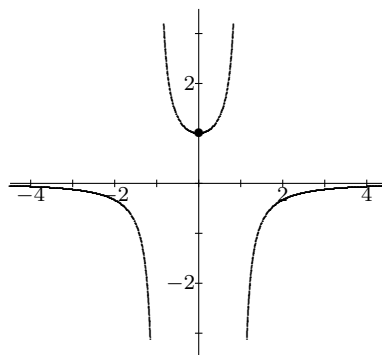
$$\begin{aligned}
 y' &= \frac{(1-x^2)(0) - (1)(-2x)}{(1-x^2)^2} \\
 &= \frac{2x}{(1-x^2)^2} \\
 y'' &= \frac{(1-x^2)^2(2) - (2x)(2(1-x^2)(-2x))}{[(1-x^2)^2]^2} \\
 &= \frac{2(1-x^2)[(1-x^2) - (2x)(-2x)]}{(1-x^2)^4} \\
 &= \frac{2(1+3x^2)}{(1-x^2)^3}
 \end{aligned}$$

y' is defined for $x \neq \pm 1$ and $y' = 0$ when $x = 0$. y'' is defined for $x \neq \pm 1$ and $y'' = 0$ when $1 + 3x^2 = 0$, which never happens. So we have the following:

interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
y'	-	-	+	+
y''	-	+	+	-
I/D	D	D	I	I
CU/CD	CD	CU	CU	CD

So we have a local min at $(0, 1)$, no local max and no inflection points.

(e) So our graph looks like



23. skip

$$24. y = \frac{1}{x} + \frac{1}{x+1} = \frac{2x+1}{x(x+1)} = \frac{2x+1}{x^2+x}$$

(a) Domain: $x \neq 0, -1$

(b) Intercepts: Since $x = 0$ is not in the domain, there is no y -intercept.

x -intercepts when $0 = \frac{2x+1}{x(x+1)}$, which happens when $x = -\frac{1}{2}$. So one x -intercept at $(-\frac{1}{2}, 0)$.

(c) Asymptotes: Vertical asymptotes: $x = 0$ and $x = -1$

$$\begin{aligned}\lim_{x \rightarrow -1^-} \frac{2x+1}{x(x+1)} &= -\infty \\ \lim_{x \rightarrow -1^+} \frac{2x+1}{x(x+1)} &= +\infty \\ \lim_{x \rightarrow 0^-} \frac{2x+1}{x(x+1)} &= -\infty \\ \lim_{x \rightarrow 0^+} \frac{2x+1}{x(x+1)} &= +\infty\end{aligned}$$

Horizontal Asymptotes:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x+1}{x^2+x} &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x}} = 0 \\ \lim_{x \rightarrow -\infty} \frac{2x+1}{x^2+x} &= \lim_{x \rightarrow -\infty} \frac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x}} = 0\end{aligned}$$

So horizontal asymptote of $y = 0$.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

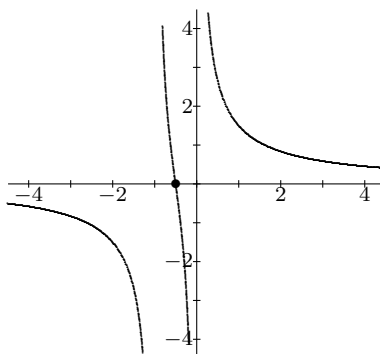
$$\begin{aligned}y' &= \frac{(x^2+x)(2) - (2x+1)(2x+1)}{(x^2+x)^2} \\ &= \frac{2x^2+2x-4x^2-4x-1}{(x^2+x)^2} \\ &= \frac{-2x^2-2x-1}{(x^2+x)^2} \\ &= \frac{-(2x^2+2x+1)}{(x^2+x)^2} \\ &= \frac{-(2x^2+2x+1)}{x^2(x+1)^2} \\ y'' &= \frac{(x^2+x)^2(-4x-2) - (-2x^2-2x-1)(2(x^2+x)(2x+1))}{[(x^2+x)^2]^2} \\ &= \frac{-2(x^2+x)^2(2x+1) + 2(2x^2+2x+1)(x^2+x)(2x+1)}{(x^2+x)^4} \\ &= \frac{2(x^2+x)(2x+1)[-(x^2+x) + (2x^2+2x+1)]}{(x^2+x)^4} \\ &= \frac{2(2x+1)(x^2+x+1)}{(x^2+x)^3} \\ &= \frac{2(2x+1)(x^2+x+1)}{x^3(x+1)^3}\end{aligned}$$

y' is defined for $x \neq 0, -1$ and $y' = 0$ when $2x^2 + 2x + 1 = 0$, which never happens.
 y'' is defined for $x \neq 0, -1$ and $y'' = 0$ when $2(2x + 1)(x^2 + x + 1) = 0$, which happens only when $2x + 1 = 0$ i.e., when $x = -\frac{1}{2}$. So we have the following:

interval	$(-\infty, -1)$	$(-1, -\frac{1}{2})$	$(-\frac{1}{2}, 0)$	$(0, \infty)$
y'	-	-	-	-
y''	-	+	-	+
I/D	D	D	D	D
CU/CD	CD	CU	CD	CU

So we have no local min, no local max and an inflection point at $(-\frac{1}{2}, 0)$.

(e) So our graph looks like



25. skip

26. $y = x + \sqrt{1-x}$

(a) Domain: $x \leq 1$

(b) Intercepts: y -intercept when $y = 0 + \sqrt{1-0} = 1$, so the y -intercept is $(0, 1)$.
 x -intercepts when

$$\begin{aligned} 0 &= x + \sqrt{1-x} \\ -x &= \sqrt{1-x} \\ (-x)^2 &= (\sqrt{1-x})^2 \\ x^2 &= 1-x \\ x^2 + x - 1 &= 0 \end{aligned}$$

This does not factor, so we use the quadratic formula to find

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Notice that $x = \frac{-1+\sqrt{5}}{2}$ is an extraneous root (is not a solution of $0 = x + \sqrt{1-x}$), so we throw it out. So we have one x -intercept at $\left(\frac{-1-\sqrt{5}}{2}, 0\right)$.

(c) Asymptotes: No vertical asymptotes

Horizontal Asymptotes: (because of the domain, we only need to look as $x \rightarrow -\infty$)

$$\begin{aligned} \lim_{x \rightarrow -\infty} (x + \sqrt{1-x}) &= \lim_{x \rightarrow -\infty} \frac{(x + \sqrt{1-x})(x - \sqrt{1-x})}{x - \sqrt{1-x}} \\ &= \lim_{x \rightarrow -\infty} \frac{x^2 - (1-x)}{x - \sqrt{1-x}} \\ &= \lim_{x \rightarrow -\infty} \frac{x^2 + x - 1}{x - \sqrt{1-x}} \\ &= \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{\frac{1}{x} - \sqrt{\frac{1}{x^4} - \frac{1}{x^3}}} \\ &= -\infty \end{aligned}$$

So no horizontal asymptote.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

$$\begin{aligned} y' &= 1 + \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \\ &= 1 - \frac{1}{2\sqrt{1-x}} \\ &= \frac{2\sqrt{1-x} - 1}{2\sqrt{1-x}} \\ y'' &= -\frac{1}{2} \left(-\frac{1}{2}\right) (1-x)^{-\frac{3}{2}}(-1) \\ &= -\frac{1}{4(1-x)^{\frac{3}{2}}} \end{aligned}$$

y' is defined for $x < 1$ and $y' = 0$ when

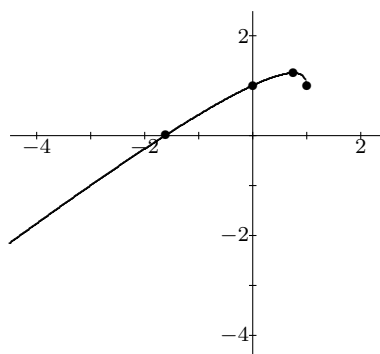
$$\begin{aligned} 2\sqrt{1-x} - 1 &= 0 \\ 2\sqrt{1-x} &= 1 \\ \sqrt{1-x} &= \frac{1}{2} \\ (\sqrt{1-x})^2 &= \left(\frac{1}{2}\right)^2 \\ 1-x &= \frac{1}{4} \\ \frac{3}{4} &= x \end{aligned}$$

y'' is defined for $x < 1$ and $y'' = 0$ when $-1 = 0$, which never happens. So we have the following:

interval	$(-\infty, \frac{3}{4})$	$(\frac{3}{4}, 1)$
y'	+	-
y''	-	-
I/D	I	D
CU/CD	CD	CD

So we have no local min, a local max at $(\frac{3}{4}, \frac{5}{4})$ and no inflection points.

(e) So our graph looks like



27. skip

28. $y = \sqrt{x} - \sqrt[3]{x} = x^{\frac{1}{2}} - x^{\frac{1}{3}}$

(a) Domain: $x \geq 0$

(b) Intercepts: y -intercept when $y = \sqrt{0} - \sqrt[3]{0} = 0$ so we have the y -intercept at $(0, 0)$.
 x -intercepts when

$$\begin{aligned} 0 &= x^{\frac{1}{2}} - x^{\frac{1}{3}} \\ x^{\frac{1}{2}} &= x^{\frac{1}{3}} \\ (x^{\frac{1}{2}})^6 &= (x^{\frac{1}{3}})^6 \\ x^3 &= x^2 \\ x^3 - x^2 &= 0 \\ x^2(x - 1) &= 0 \end{aligned}$$

which happens when $x = 0$ and $x = 1$. So two x -intercepts at $(0, 0)$ and $(1, 0)$.

(c) Asymptotes: No vertical asymptotes.

Horizontal Asymptotes (because of the domain, we only need to look as $x \rightarrow \infty$):

$$\begin{aligned} \lim_{x \rightarrow \infty} (x^{\frac{1}{2}} - x^{\frac{1}{3}}) &= \lim_{x \rightarrow \infty} (x^{\frac{3}{6}} - x^{\frac{2}{6}}) \\ &= \lim_{x \rightarrow \infty} x^{\frac{2}{6}} (x^{\frac{1}{6}} - 1) \\ &= \lim_{x \rightarrow \infty} x^{\frac{1}{3}} (x^{\frac{1}{6}} - 1) \\ &= \infty \end{aligned}$$

So no horizontal asymptote.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

$$\begin{aligned} y' &= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{3x^{\frac{2}{3}}} \\ &= \frac{1}{2x^{\frac{3}{6}}} - \frac{1}{3x^{\frac{4}{6}}} \\ &= \frac{3x^{\frac{1}{6}}}{6x^{\frac{4}{6}}} - \frac{2}{6x^{\frac{4}{6}}} \\ &= \frac{3x^{\frac{1}{6}} - 2}{6x^{\frac{2}{3}}} \\ y'' &= \frac{1}{2} \left(-\frac{1}{2} \right) x^{-\frac{3}{2}} - \frac{1}{3} \left(-\frac{2}{3} \right) x^{-\frac{5}{3}} \\ &= -\frac{1}{4x^{\frac{3}{2}}} + \frac{2}{9x^{\frac{5}{3}}} \\ &= -\frac{1}{4x^{\frac{9}{6}}} + \frac{2}{9x^{\frac{10}{6}}} \\ &= -\frac{9x^{\frac{1}{6}}}{36x^{\frac{10}{6}}} + \frac{8}{36x^{\frac{10}{6}}} \\ &= \frac{-9x^{\frac{1}{6}} + 8}{36x^{\frac{5}{3}}} \end{aligned}$$

y' is defined for $x \neq 0$ and $y' = 0$ when

$$\begin{aligned} 3x^{\frac{1}{6}} - 2 &= 0 \\ 3x^{\frac{1}{6}} &= 2 \\ x^{\frac{1}{6}} &= \frac{2}{3} \\ (x^{\frac{1}{6}})^6 &= \left(\frac{2}{3} \right)^6 \\ x &= \frac{64}{729} \end{aligned}$$

y'' is defined for $x \neq 0$ and $y'' = 0$ when

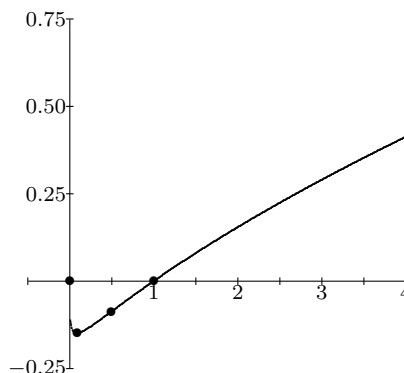
$$\begin{aligned} -9x^{\frac{1}{6}} + 8 &= 0 \\ 8 &= 9x^{\frac{1}{6}} \\ \frac{8}{9} &= x^{\frac{1}{6}} \\ \left(\frac{8}{9}\right)^6 &= (x^{\frac{1}{6}})^6 \\ \left(\frac{8}{9}\right)^6 &= x \end{aligned}$$

So we have the following:

interval	$(0, \frac{64}{729})$	$(\frac{64}{729}, (\frac{8}{9})^6)$	$(\frac{8}{9})^6, \infty)$
y'	-	+	+
y''	+	+	-
I/D	D	I	D
CU/CD	CU	CU	CD

So we have a local min at $(\frac{64}{729}, -\frac{4}{27})$, no local max and an inflection point at $(\frac{8}{9})^6, -\frac{64}{729}$.

(e) So our graph looks like



29. skip

30. $y = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(a) Domain: given as $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) Intercepts: y -intercept when $y = 4(0) - \tan(0) = 0$, so the y -intercept is $(0, 0)$.
 x -intercepts (using calculator) at $(-1.39, 0)$, $(0, 0)$ and $(1.39, 0)$.

(c) Asymptotes: Vertical asymptotes: $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$

$$\begin{aligned}
 \lim_{x \rightarrow -\frac{\pi}{2}^+} (4x - \tan x) &= \lim_{x \rightarrow -\frac{\pi}{2}^+} 4x - \lim_{x \rightarrow -\frac{\pi}{2}^+} \tan x \\
 &= -2\pi - (-\infty) \\
 &= \infty \\
 \lim_{x \rightarrow \frac{\pi}{2}^-} (4x - \tan x) &= \lim_{x \rightarrow \frac{\pi}{2}^-} 4x - \lim_{x \rightarrow \frac{\pi}{2}^-} \tan x \\
 &= 2\pi - (\infty) \\
 &= -\infty
 \end{aligned}$$

Horizontal Asymptotes: not necessary due to the domain.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

$$\begin{aligned}
 y' &= 4 - \sec^2 x \\
 &= 4 - \frac{1}{\cos^2 x} \\
 &= \frac{4\cos^2 x - 1}{\cos^2 x} \\
 y'' &= -2\sec x(\sec x \tan x) \\
 &= -2\sec^2 x \tan x \\
 &= -\frac{2\sin x}{\cos^3 x}
 \end{aligned}$$

y' is not defined for

$$\begin{aligned}
 \cos^2 x &= 0 \\
 \cos x &= 0 \\
 x &= \pm \frac{\pi}{2} + 2k\pi
 \end{aligned}$$

which never happens on the domain. $y' = 0$ when

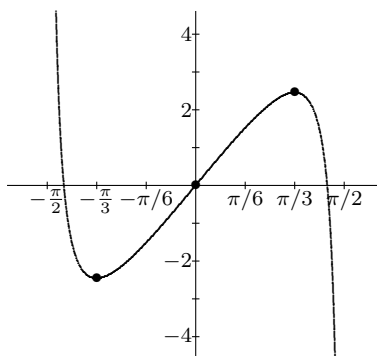
$$\begin{aligned}
 4\cos^2 x - 1 &= 0 \\
 4\cos^2 x &= 1 \\
 \cos^2 x &= \frac{1}{4} \\
 \sqrt{\cos^2 x} &= \pm \sqrt{\frac{1}{4}} \\
 \cos x &= \pm \frac{1}{2}
 \end{aligned}$$

But $\cos x$ is positive on the domain of the function, so we only need to find where $\cos x = \frac{1}{2}$, which is when $x = \pm \frac{\pi}{3}$. y'' is not defined for $\cos^3 x = 0$, which never happens on the domain. $y'' = 0$ when $-2\sin x = 0$, which happens on the domain only when $x = 0$. So we have the following:

interval	$(-\frac{\pi}{2}, -\frac{\pi}{3})$	$(-\frac{\pi}{3}, 0)$	$(0, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{\pi}{2})$
y'	-	+	+	-
y''	+	+	-	-
I/D	D	I	I	D
CU/CD	CU	CU	CD	CD

So we have a local min at $(-\frac{\pi}{3}, \sqrt{3} - \frac{4\pi}{3})$, a local max at $(\frac{\pi}{3}, \frac{4\pi}{3} - \sqrt{3})$ and an inflection point at $(0, 0)$.

(e) So our graph looks like



31. skip

32. $y = e^{2x-x^2} = e^{x(2-x)}$

(a) Domain: all real numbers

(b) Intercepts: y -intercept when $y = e^{2(0)-(0)^2} = e^0 = 1$, so the y -intercept is $(0, 1)$.
 x -intercepts when $0 = e^{2x-x^2}$, which never happens. So no x -intercepts.

(c) Asymptotes: No vertical asymptotes.

Horizontal Asymptotes:

$$\lim_{x \rightarrow \infty} e^{x(2-x)} = 0$$

$$\lim_{x \rightarrow -\infty} e^{x(2-x)} = 0$$

(in both cases you have e raised to a large, negative exponent) So the horizontal asymptote is $y = 0$.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

$$\begin{aligned}
 y' &= (2 - 2x)e^{2x-x^2} \\
 &= 2(1 - x)e^{2x-x^2} \\
 y'' &= (2 - 2x)[(2 - 2x)e^{2x-x^2}] + (-2)e^{2x-x^2} \\
 &= [(2 - 2x)(2 - 2x) - 2]e^{2x-x^2} \\
 &= (4x^2 - 8x + 2)e^{2x-x^2} \\
 &= 2(2x^2 - 4x + 1)e^{2x-x^2}
 \end{aligned}$$

y' is defined everywhere. $y' = 0$ when

$$\begin{aligned}
 2(1 - x)e^{2x-x^2} &= 0 \\
 1 - x &= 0 \\
 1 &= x
 \end{aligned}$$

y'' is defined everywhere. $y'' = 0$ when $2x^2 - 4x + 1 = 0$. This doesn't factor, so we use the quadratic formula

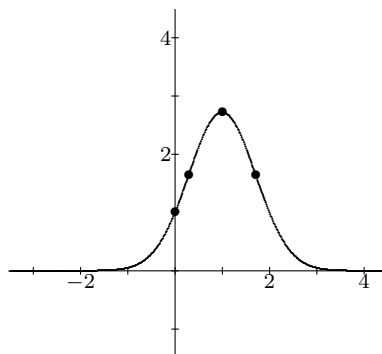
$$\begin{aligned}
 x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)} \\
 &= \frac{4 \pm \sqrt{8}}{4} \\
 &= \frac{4 \pm 2\sqrt{2}}{4} \\
 &= 1 \pm \frac{\sqrt{2}}{2}
 \end{aligned}$$

So we have the following:

interval	$(-\infty, 1 - \frac{\sqrt{2}}{2})$	$(1 - \frac{\sqrt{2}}{2}, 1)$	$(1, 1 + \frac{\sqrt{2}}{2})$	$(1 + \frac{\sqrt{2}}{2}, \infty)$
y'	+	+	-	-
y''	+	-	-	+
I/D	I	I	D	D
CU/CD	CU	CD	CD	CU

So we have no local min, a local max at $(1, e)$ and inflection points at $(1 - \frac{\sqrt{2}}{2}, e^{\frac{1}{2}})$ and $(1 + \frac{\sqrt{2}}{2}, e^{\frac{1}{2}})$.

(e) So our graph looks like



33. skip

34. $y = \ln(x^2 - 1)$

- (a) Domain: $x^2 - 1 > 0$ which happens when $x > 1$ and $x < -1$
 (b) Intercepts: $x = 0$ is not in the domain, so there is no y -intercept.
 x -intercepts when

$$\begin{aligned} \ln(x^2 - 1) &= 0 \\ e^{\ln(x^2 - 1)} &= e^0 \\ x^2 - 1 &= 1 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

So two x -intercepts at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.

- (c) Asymptotes: Vertical asymptote of the log function happens when the argument approaches 0 from above. $x^2 - 1 \rightarrow 0^+$ when $x \rightarrow 1^+$ and $x \rightarrow -1^-$. So we have

$$\begin{aligned} \lim_{x \rightarrow 1^+} \ln(x^2 - 1) &= -\infty \\ \lim_{x \rightarrow -1^-} \ln(x^2 - 1) &= -\infty \end{aligned}$$

Horizontal Asymptotes:

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln(x^2 - 1) &= \infty \\ \lim_{x \rightarrow -\infty} \ln(x^2 - 1) &= \infty \end{aligned}$$

So no horizontal asymptote.

(d) Intervals of I/D, local extrema, intervals of concavity and inflection points:

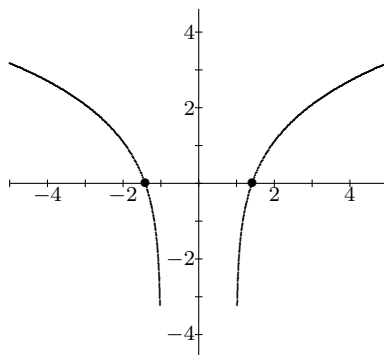
$$\begin{aligned}
 y' &= \frac{2x}{x^2 - 1} \\
 &= \frac{2x}{(x-1)(x+1)} \\
 y'' &= \frac{(x^2 - 1)(2) - (2x)(2x)}{(x^2 - 1)^2} \\
 &= \frac{-2x^2 - 2}{(x-1)^2(x+1)^2} \\
 &= \frac{-2(x^2 + 1)}{(x-1)^2(x+1)^2}
 \end{aligned}$$

y' is defined for $x \neq \pm 1$. $y' = 0$ when $x = 0$. y'' is defined for $x \neq \pm 1$. $y'' = 0$ when $-2(x^2 + 1) = 0$, which never happens. So we have the following:

interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
y'	-	NOT DEFINED	+
y''	-	NOT DEFINED	-
I/D	D		I
CU/CD	CD		CD

So we have no local min, no local max and no inflection points.

(e) So our graph looks like



35.-44. skip

45. Show that the equation $x^{101} + x^{51} + x - 1 = 0$ has exactly one real root.

Let $f(x) = x^{101} + x^{51} + x - 1$. Notice that $f(0) = -1$ and $f(1) = 2$, so by the Intermediate Value Theorem, there is at least one root of f in the interval $[0, 1]$.

If we look at values of $x < 0$, then x to an odd power is negative, so each term of f is negative. Thus the sum of a set of negative numbers will be negative and we have that

when $x < 0$, $f(x) < 0$. This tells us that there are no roots of f less than 0.

Similarly, if we look at values of $x > 1$, then x to an odd power is positive and our function is the sum of three numbers bigger than one minus one. Thus, when $x > 1$, $f(x) > 0$. This tells us there are no roots of f greater than 1.

So all of the roots of f must occur between 0 and 1.

Suppose there is more than one root of f on $[0, 1]$, say $f(a) = f(b) = 0$ for $0 < a, b < 1$. Notice that f is continuous and differentiable on $[0, 1]$, so it is continuous and differentiable on $[a, b]$. So by Rolle's Theorem, we know that there is a number c , $a < c < b$ so that $f'(c) = 0$.

But $f'(x) = 101x^{100} + 51x^{50} + 1$, so if $f'(c) = 0$, we have

$$\begin{aligned} 101c^{100} + 51c^{50} + 1 &= 0 \\ 101c^{100} + 51c^{50} &= -1 \end{aligned}$$

But the left side of the equation is always positive, so we have a contradiction. Thus, our assumption that there is more than one root of f on $[0, 1]$ is false. Since there is at least one and not more than one, there is exactly one root on $[0, 1]$.

46. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leq f'(x) \leq 5$ for all x in $(0, 4)$. Show that $9 \leq f(4) \leq 21$.

We will apply the Mean Value Theorem to f on $[0, 4]$.

We are told that f is continuous on $[0, 4]$ and the statement that $2 \leq f'(x) \leq 5$ for all x in $(0, 4)$ implies that f is differentiable on $(0, 4)$. So by the MVT, there exists a number c , $0 < c < 4$ so that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - 1}{4}$$

Since

$$2 \leq f'(x) \leq 5$$

on $(0, 4)$, we know that

$$\begin{aligned} 2 &< \frac{f(4) - 1}{4} < 5 \\ 8 &< f(4) - 1 < 20 \\ 9 &< f(4) < 21 \end{aligned}$$

47. By applying the Mean Value Theorem to the function $f(x) = x^{\frac{1}{5}}$ on the interval $[32, 33]$, show that

$$2 \leq \sqrt[5]{33} < 2.0125$$

Notice that f is continuous on $[32, 33]$ and differentiable on $(32, 33)$ because root functions are continuous and differentiable on their domains. So we apply the MVT and there exists a number c , $32 < c < 33$ so that

$$\begin{aligned} f'(c) &= \frac{f(33) - f(32)}{33 - 32} \\ &= \frac{\sqrt[5]{33} - \sqrt[5]{32}}{1} \\ &= \sqrt[5]{33} - 2 \end{aligned}$$

Notice that

$$f'(x) = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5\sqrt[5]{x^4}}$$

so $f'(c) > 0$ and we have

$$\begin{aligned} \sqrt[5]{33} - 2 &> 0 \\ \sqrt[5]{33} &> 2 \end{aligned}$$

To show the other half of the inequality, we look at

$$f''(x) = -\frac{4}{25}x^{-\frac{9}{5}} = -\frac{4}{25\sqrt[5]{x^9}}$$

which is negative for x on the interval $[32, 33]$. This means that f' is decreasing on $[32, 33]$ and since $c > 32$, we have

$$f'(c) < f'(32) = \frac{1}{5\sqrt[5]{32^4}} = \frac{1}{80} = 0.0125$$

Thus

$$\begin{aligned} \sqrt[5]{33} - 2 &< 0.0125 \\ \sqrt[5]{33} &< 2.0125 \end{aligned}$$

So the two inequalities give us

$$2 < \sqrt[5]{33} < 2.0125$$

48. For what values of the constants a and b is $(1, 6)$ a point of inflection of the curve $y = x^3 + ax^2 + bx + 1$?

If $(1, 6)$ is a point of inflection, it lies on the curve and hence

$$6 = (1)^3 + a(1)^2 + b(1) + 1$$

which gives us

$$a + b = 4$$

To be a point of inflection, the function must be continuous at that point and change concavity at that point (i.e. the second derivative changes sign at that point). So the second derivative must be zero at that point. (Having the second derivative be zero is not enough, we must also check that the second derivative changes sign there)

$$\begin{aligned}y' &= 3x^2 + 2ax + b \\y'' &= 6x + 2a\end{aligned}$$

If $(1, 6)$ is an inflection point, we have

$$\begin{aligned}0 &= 6(1) + 2a \\-6 &= 2a \\-3 &= a\end{aligned}$$

So $-3 + b = 4$, which means $b = 7$. So we have

$$y = x^3 - 3x^2 + 7x + 1$$

and

$$y'' = 6x - 6 = 6(x - 1)$$

Notice that when $x < 1$, $y'' < 0$ and when $x > 1$, $y'' > 0$, so the second derivative does in fact change sign when $x = 1$, and $(1, 6)$ is an inflection point of $y = x^3 - 3x^2 + 7x + 1$.