16.

$$\begin{aligned} y &= \ln(\csc 5x) \\ y' &= \frac{\frac{d}{dx}(\csc 5x)}{\csc 5x} \\ &= \frac{-5\csc 5x \cot 5x}{\csc 5x} \\ &= -5\cot 5x \end{aligned}$$

20.

$$\begin{aligned} y &= \ln(x^2 e^x) \\ y' &= \frac{\frac{d}{dx}(x^2 e^x)}{x^2 e^x} \\ &= \frac{x^2 e^x + 2xe^x}{x^2 e^x} \\ &= \frac{xe^x(x+2)}{x^2 e^x} \\ &= \frac{x+2}{x} \end{aligned}$$

27.

$$\begin{aligned} y &= \log_5(1+2x) \\ y' &= \frac{\frac{d}{dx}(1+2x)}{(1+2x)(\ln 5)} \\ &= \frac{2}{(1+2x)(\ln 5)} \end{aligned}$$

28.

$$\begin{aligned} y &= (\cos x)^x \\ \ln y &= \ln(\cos x)^x \\ &= x\ln(\cos x) \\ \frac{y'}{y} &= x\left(\frac{\frac{d}{dx}(\cos x)}{\cos x}\right) + \ln(\cos x) \\ \frac{y'}{y} &= x\left(\frac{-\sin x}{\cos x}\right) + \ln(\cos x) \\ \frac{y'}{y} &= -x\tan x + \ln(\cos x) \\ y' &= y(-x\tan x + \ln(\cos x)) \\ &= (\cos x)^x(-x\tan x + \ln(\cos x)) \end{aligned}$$

29.

$$\begin{aligned}
y &= \ln \sin x - \frac{1}{2}\sin^2 x \\
y' &= \frac{\frac{d}{dx}(\sin x)}{\sin x} - \frac{1}{2}(2\sin x \cos x) \\
&= \frac{\cos x}{\sin x} - \sin x \cos x \\
&= \cot x - \sin x \cos x
\end{aligned}$$

30.

$$\begin{aligned}
y &= \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \\
\ln y &= \ln\left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\right) \\
&= \ln(x^2+1)^4 - \ln(2x+1)^3(3x-1)^5 \\
&= \ln(x^2+1)^4 - \ln(2x+1)^3 - \ln(3x-1)^5 \\
&= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \\
\frac{y'}{y} &= 4\left(\frac{\frac{d}{dx}(x^2+1)}{x^2+1}\right) - 3\left(\frac{\frac{d}{dx}(2x+1)}{2x+1}\right) - 5\left(\frac{\frac{d}{dx}(3x-1)}{3x-1}\right) \\
\frac{y'}{y} &= 4\left(\frac{2x}{x^2+1}\right) - 3\left(\frac{2}{2x+1}\right) - 5\left(\frac{3}{3x-1}\right) \\
\frac{y'}{y} &= \frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1} \\
y' &= y\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right) \\
&= \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right)
\end{aligned}$$

33.

$$\begin{aligned}
y &= \ln|\sec 5x + \tan 5x| \\
y' &= \frac{\frac{d}{dx}(\sec 5x + \tan 5x)}{\sec 5x + \tan 5x} \\
&= \frac{5\sec 5x \tan 5x + 5\sec^2 5x}{\sec 5x + \tan 5x} \\
&= \frac{5\sec 5x(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} \\
&= 5\sec 5x
\end{aligned}$$

36.

$$\begin{aligned}
y &= \sqrt{t\ln(t^4)} \\
y' &= \frac{1}{2}(t\ln(t^4))^{-\frac{1}{2}}\frac{d}{dt}(t\ln(t^4)) \\
&= \frac{1}{2}(t\ln(t^4))^{-\frac{1}{2}}\left(t\left(\frac{4t^3}{t^4}\right)+\ln(t^4)\right) \\
&= \frac{1}{2\sqrt{t\ln(t^4)}}\left(4+\ln(t^4)\right) \\
&= \frac{4+\ln(t^4)}{2\sqrt{t\ln(t^4)}}
\end{aligned}$$

41.

$$\begin{aligned}
y &= \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7} \\
\ln y &= \ln\left(\frac{(x+1)^{\frac{1}{2}}(2-x)^5}{(x+3)^7}\right) \\
&= \ln(x+1)^{\frac{1}{2}}(2-x)^5 - \ln(x+3)^7 \\
&= \ln(x+1)^{\frac{1}{2}} + \ln(2-x)^5 - \ln(x+3)^7 \\
&= \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \\
\frac{y'}{y} &= \frac{1}{2}\left(\frac{\frac{d}{dx}(x+1)}{x+1}\right) + 5\left(\frac{\frac{d}{dx}(2-x)}{2-x}\right) - 7\left(\frac{\frac{d}{dx}(x+3)}{x+3}\right) \\
\frac{y'}{y} &= \frac{1}{2}\left(\frac{1}{x+1}\right) + 5\left(\frac{-1}{2-x}\right) - 7\left(\frac{1}{x+3}\right) \\
\frac{y'}{y} &= \frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3} \\
y' &= y\left(\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3}\right) \\
&= \frac{(x+1)^{\frac{1}{2}}(2-x)^5}{(x+3)^7}\left(\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3}\right)
\end{aligned}$$

46.

$$\begin{aligned}
y &= \ln\left|\frac{x^2-4}{2x+5}\right| \\
&= \ln|x^2-4| - \ln|2x+5| \\
y' &= \frac{\frac{d}{dx}(x^2-4)}{x^2-4} - \frac{\frac{d}{dx}(2x+5)}{2x+5} \\
&= \frac{2x}{x^2-4} - \frac{2}{2x+5}
\end{aligned}$$

60.

$$\begin{aligned}
y &= xe^{\sin x} \\
\ln y &= \ln xe^{\sin x} \\
&= \ln x + \ln e^{\sin x} \\
&= \ln x + \sin x \\
\frac{y'}{y} &= \frac{1}{x} + \cos x \\
y' &= y\left(\frac{1}{x} + \cos x\right) \\
&= xe^{\sin x}\left(\frac{1}{x} + \cos x\right) \\
&= e^{\sin x}(1 + x\cos x)
\end{aligned}$$

65. If $f(x) = (x-a)(x-b)(x-c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

Let $y = f(x)$. Then

$$\begin{aligned}
y &= (x-a)(x-b)(x-c) \\
\ln y &= \ln(x-a)(x-b)(x-c) \\
&= \ln(x-a) + \ln(x-b) + \ln(x-c) \\
\frac{y'}{y} &= \frac{\frac{d}{dx}(x-a)}{x-a} + \frac{\frac{d}{dx}(x-b)}{x-b} + \frac{\frac{d}{dx}(x-c)}{x-c} \\
&= \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}
\end{aligned}$$

75. Find f' in terms of g': $f(x) = \ln|g(x)|$

$$f'(x) = \frac{g'(x)}{g(x)}$$

76. Find f' in terms of g': $f(x) = g(\ln x)$

$$f'(x) = g'(\ln x)\frac{d}{dx}(\ln x) = \frac{g'(\ln x)}{x}$$

81. At what point on the curve $y = [\ln(x+4)]^2$ is the tangent horizontal?

$$\begin{aligned} y &= [\ln(x+4)]^2 \\ y' &= 2[\ln(x+4)]\frac{d}{dx}[\ln(x+4)] \\ &= 2\ln(x+4)\left(\frac{\frac{d}{dx}(x+4)}{x+4}\right) \\ &= 2\ln(x+4)\left(\frac{1}{x+4}\right) \\ &= \frac{2\ln(x+4)}{x+4} \end{aligned}$$

To find where the tangent is horizontal, set $y' = 0$ and solve for x:

$$\begin{aligned} \frac{2\ln(x+4)}{x+4} &= 0 \\ 2\ln(x+4) &= 0 \\ \ln(x+4) &= 0 \\ e^{\ln(x+4)} &= e^0 \\ x+4 &= 1 \\ x &= -3 \end{aligned}$$

If $x = -3$, then

$$y = [\ln(-3+4)]^2 = [\ln(1)]^2 = 0^2 = 0$$

so the tangent is horizontal at the point $(-3, 0)$.

88. The volume of a right circular cone is $V = \dfrac{\pi r^2 h}{3}$, where r is the radius of the base and h is the height.

1. Find the rate of change of the volume with respect to the height if the radius is constant.

$$\frac{dV}{dt} = \frac{\pi r^2}{3}\frac{dh}{dt}$$

2. Find the rate of change of the volume with respect to the radius if the height is constant.

$$\frac{dV}{dt} = \frac{\pi h(2r)}{3}\frac{dr}{dt} = \frac{2\pi rh}{3}\frac{dr}{dt}$$

91. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of the edge is 30 cm?

Let x=edge of the cube, V=volume of the cube, A=surface area of the cube. We are told that $\dfrac{dV}{dt} = 10$ cm^3/min. We want to find $\dfrac{dA}{dt}$ when $x = 30$ cm. We also have the relationships

$$V = x^3 \qquad A = 6x^2$$

We can use these two equations to relate V and A. If we solve each equation for x, we have

$$x = V^{\frac{1}{3}} \qquad x = \frac{A^{\frac{1}{2}}}{6^{\frac{1}{2}}}$$

So we have the relationship

$$V^{\frac{1}{3}} = \frac{A^{\frac{1}{2}}}{6^{\frac{1}{2}}}$$

which can be written as

$$6^{\frac{1}{2}} V^{\frac{1}{3}} = A^{\frac{1}{2}}$$

So we have

$$\begin{aligned} \frac{1}{2}A^{-\frac{1}{2}}\frac{dA}{dt} &= 6^{\frac{1}{2}}\frac{1}{3}V^{-\frac{2}{3}}\frac{dV}{dt} \\ \frac{1}{2}A^{-\frac{1}{2}}\frac{dA}{dt} &= \frac{6^{\frac{1}{2}}}{3}V^{-\frac{2}{3}}\frac{dV}{dt} \end{aligned}$$

When $x = 30$ cm, we have $V = 30^3 = 27000\ \text{cm}^3$ and $A = 6(30)^2 = 5400\ \text{cm}^2$, so we have

$$\begin{aligned} \frac{1}{2}(5400)^{-\frac{1}{2}}\frac{dA}{dt} &= \frac{6^{\frac{1}{2}}}{3}(27000)^{-\frac{2}{3}}(10) \\ \frac{dA}{dt} &= \left(\frac{10 \cdot 6^{\frac{1}{2}}}{3(27000)^{\frac{2}{3}}}\right)\left(2(5400)^{\frac{1}{2}}\right) \\ &= \frac{4}{3}\text{cm}^2/\text{min} \end{aligned}$$

92. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

Let V=volume of water in the cup, h=height of water in the cup, and r=radius at the water level. So we know that $\frac{dV}{dt} = 2\ \text{cm}^3/\text{s}$ and we want to find $\frac{dh}{dt}$ when $h = 5$ cm. We can relate V, h and r by

$$V = \frac{1}{3}\pi r^2 h$$

but we need to relate V to h alone. We can find r in terms of h by using similar triangles

$$\frac{r}{3} = \frac{h}{10}$$

so $r = \frac{3h}{10}$ and our relationship is now

$$V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100}h^3$$

and so

$$\frac{dV}{dt} = \frac{3\pi}{100}3h^2\frac{dh}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt}$$

So we have

$$\begin{aligned} 2 &= \frac{9\pi(5)^2}{100}\frac{dh}{dt} \\ \frac{8}{9\pi} &= \frac{dh}{dt} \end{aligned}$$

93. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?
Let x=horizontal distance the boy has traveled, y=height of the balloon, and z=distance between the boy and the balloon. We know that

$$\frac{dx}{dt} = 15 \text{ ft/s} \qquad \frac{dy}{dt} = 5 \text{ ft/s}$$

and we want to know $\frac{dz}{dt}$ after 3 seconds, that is when $x = 15(3) = 45$ and $y = 45+5(3) = 60$
We know that

$$x^2 + y^2 = z^2$$

so we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

When $x = 45$ and $y = 60$ we have $z^2 = 45^2 + 60^2 = 5625$ so $z = 75$. That gives us

$$\begin{aligned} 2(45)(15) + 2(60)(5) &= 2(75)\frac{dz}{dt} \\ 1950 &= 150\frac{dz}{dt} \\ 13 \text{ ft/s} &= \frac{dz}{dt} \end{aligned}$$