Lenarz | Practice Exam #2 | Name: ______________
Math 121 | October 27, 2006 | Section: ______________

Directions: Answer the following questions on a separate piece of paper. You may not use a calculator. If you do not show your work, you will receive no credit. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is much longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Calculate y'

   (a) $y = (x^4 - 3x^2 + 5)^3$

   Solution:

   $$\begin{aligned} y' &= 3(x^4 - 3x^2 + 5)^2 \frac{d}{dx}(x^4 - 3x^2 + 5) \\ &= 3(x^4 - 3x^2 + 5)^2 (4x^3 - 6x) \end{aligned}$$

   (b) $y = \cos(\tan x)$

   Solution:

   $$\begin{aligned} y' &= -\sin(\tan x) \frac{d}{dx}(\tan x) \\ &= -\sin(\tan x) \sec^2 x \end{aligned}$$

   (c) $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$

   Solution:

   $$\begin{aligned} y &= x^{1/2} + x^{-4/3} \\ y' &= \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} \\ &= \frac{1}{2\sqrt{x}} - \frac{4}{3\sqrt[3]{x^7}} \end{aligned}$$

(d) $y = \dfrac{3x-2}{\sqrt{2x+1}}$

Solution:

$$
\begin{aligned}
y &= \frac{3x-2}{(2x+1)^{1/2}} \\
y' &= \frac{(2x+1)^{1/2}\frac{d}{dx}(3x-2) - (3x-2)\frac{d}{dx}[(2x+1)^{1/2}]}{\left((2x+1)^{1/2}\right)^2} \\
&= \frac{(2x+1)^{1/2}(3) - (3x-2)\left[\frac{1}{2}(2x+1)^{-1/2}\frac{d}{dx}(2x+1)\right]}{2x+1} \\
&= \frac{3(2x+1)^{1/2} - (3x-2)\left[\frac{1}{2}(2x+1)^{-1/2}(2)\right]}{2x+1} \\
&= \frac{3(2x+1)^{1/2} - (3x-2)(2x+1)^{-1/2}}{2x+1} \\
&= \frac{3(2x+1)^{1/2} - (3x-2)(2x+1)^{-1/2}}{2x+1}\left(\frac{(2x+1)^{1/2}}{(2x+1)^{1/2}}\right) \\
&= \frac{3(2x+1) - (3x-2)}{(2x+1)^{3/2}} \\
&= \frac{6x+3-3x+2}{(2x+1)^{3/2}} \\
&= \frac{3x+5}{(2x+1)^{3/2}}
\end{aligned}
$$

(e) $y = 2x\sqrt{x^2+1}$

Solution:

$$\begin{aligned} y &= 2x(x^2+1)^{1/2} \\ y' &= 2x\frac{d}{dx}[(x^2+1)^{1/2}] + (x^2+1)^{1/2}\frac{d}{dx}(2x) \\ &= 2x\left[\frac{1}{2}(x^2+1)^{-1/2}\frac{d}{dx}(x^2+1)\right] + (x^2+1)^{1/2}(2) \\ &= x[(x^2+1)^{-1/2}(2x)] + 2(x^2+1)^{1/2} \\ &= 2x^2(x^2+1)^{-1/2} + 2(x^2+1)^{1/2} \\ &= \frac{2x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2+1} \\ &= \frac{4x^2+2}{\sqrt{x^2+1}} \end{aligned}$$

(f) $y = e^{\sin 2\theta}$

Solution:

$$\begin{aligned} y' &= e^{\sin 2\theta}\frac{d}{d\theta}(\sin 2\theta) \\ &= e^{\sin 2\theta}\left(\cos 2\theta\frac{d}{d\theta}(2\theta)\right) \\ &= e^{\sin 2\theta}(\cos 2\theta)(2) \\ &= (2\cos 2\theta)e^{\sin 2\theta} \end{aligned}$$

(g) $y = \sin^{-1}(e^x)$

Solution:

$$\begin{aligned} y' &= \frac{1}{\sqrt{1-(e^x)^2}}\frac{d}{dx}(e^x) \\ &= \frac{1}{\sqrt{1-e^{2x}}}(e^x) \\ &= \frac{e^x}{\sqrt{1-e^{2x}}} \end{aligned}$$

(h) $y = xe^{-1/x}$

Solution:

$$
\begin{aligned}
y' &= x\frac{d}{dx}(e^{-1/x}) + (e^{-1/x})\frac{d}{dx}(x) \\
&= x\left(e^{-1/x}\frac{d}{dx}\left(-\frac{1}{x}\right)\right) + (e^{-1/x})(1) \\
&= x\left(e^{-1/x}\frac{d}{dx}\left(-x^{-1}\right)\right) + e^{-1/x} \\
&= x\left(e^{-1/x}\left(x^{-2}\right)\right) + e^{-1/x} \\
&= x^{-1}e^{-1/x} + e^{-1/x} \\
&= e^{-1/x}(x^{-1}+1) \\
&= e^{-1/x}\left(\frac{1}{x}+1\right)
\end{aligned}
$$

(i) $y = \ln(\csc 5x)$

Solution:

$$
\begin{aligned}
y' &= \frac{\frac{d}{dx}(\csc 5x)}{\csc 5x} \\
&= \frac{-\csc 5x \cot 5x \frac{d}{dx}(5x)}{\csc 5x} \\
&= \frac{-5\csc 5x \cot 5x}{\csc 5x} \\
&= -5\cot 5x
\end{aligned}
$$

(j) $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta}$

Solution:

$$
\begin{aligned}
y' &= \frac{(1+\tan 2\theta)\dfrac{d}{d\theta}(\sec 2\theta) - (\sec 2\theta)\dfrac{d}{d\theta}(1+\tan 2\theta)}{(1+\tan 2\theta)^2} \\
&= \frac{(1+\tan 2\theta)\left(\sec 2\theta \tan 2\theta \dfrac{d}{d\theta}(2\theta)\right) - (\sec 2\theta)\left(\sec^2 2\theta \dfrac{d}{d\theta}(2\theta)\right)}{(1+\tan 2\theta)^2} \\
&= \frac{(1+\tan 2\theta)(\sec 2\theta \tan 2\theta(2)) - (\sec 2\theta)(\sec^2 2\theta(2))}{(1+\tan 2\theta)^2} \\
&= \frac{2\sec 2\theta \tan 2\theta + 2\sec 2\theta \tan^2 2\theta - 2\sec^3 2\theta}{(1+\tan 2\theta)^2} \\
&= \frac{2\sec 2\theta(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1+\tan 2\theta)^2} \\
&= \frac{2\sec 2\theta(\tan 2\theta - 1)}{(1+\tan 2\theta)^2}
\end{aligned}
$$

The last step uses the fact that $\tan^2\theta + 1 = \sec^2\theta$.

(k) $y = \ln(x^2 e^x)$

Solution:

$$
\begin{aligned}
y' &= \frac{\frac{d}{dx}(x^2 e^x)}{x^2 e^x} \\
&= \frac{x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)}{x^2 e^x} \\
&= \frac{x^2 e^x + 2x e^x}{x^2 e^x} \\
&= \frac{x e^x (x+2)}{x^2 e^x} \\
&= \frac{x+2}{x} \\
&= 1 + \frac{2}{x}
\end{aligned}
$$

(l) $y = e^{e^x}$

Solution:

$$\begin{aligned} y' &= e^{e^x}\frac{d}{dx}(e^x) \\ &= e^{e^x}e^x \\ &= e^{e^x+x} \end{aligned}$$

(m) $y = \sec(1+x^2)$

Solution:

$$\begin{aligned} y' &= \sec(1+x^2)\tan(1+x^2)\frac{d}{dx}(1+x^2) \\ &= \sec(1+x^2)\tan(1+x^2)(2x) \\ &= 2x\sec(1+x^2)\tan(1+x^2) \end{aligned}$$

(n) $y = \log_5(1+2x)$

Solution:

$$\begin{aligned} y' &= \frac{\frac{d}{dx}(1+2x)}{(1+2x)(\ln 5)} \\ &= \frac{2}{(1+2x)(\ln 5)} \end{aligned}$$

(o) $y = \ln\sin x - \dfrac{1}{2}\sin^2 x$

Solution:

$$\begin{aligned} y' &= \frac{\frac{d}{dx}(\sin x)}{\sin x} - \frac{1}{2}\left(2\sin x\frac{d}{dx}(\sin x)\right) \\ &= \frac{\cos x}{\sin x} - \sin x\cos x \\ &= \cot x - \sin x\cos x \end{aligned}$$

(p) $y = x\tan^{-1}(4x)$

Solution:

$$\begin{aligned} y' &= x\frac{d}{dx}[\tan^{-1}(4x)] + \tan^{-1}(4x)\frac{d}{dx}(x) \\ &= x\left(\frac{1}{1+(4x)^2}\right)\frac{d}{dx}(4x) + \tan^{-1}(4x)(1) \\ &= x\left(\frac{1}{1+16x^2}\right)(4) + \tan^{-1}(4x) \\ &= \frac{4x}{1+16x^2} + \tan^{-1}(4x) \end{aligned}$$

(q) $y = \ln|\sec 5x + \tan 5x|$

Solution:

$$\begin{aligned} y' &= \frac{\frac{d}{dx}(\sec 5x + \tan 5x)}{\sec 5x + \tan 5x} \\ &= \frac{\sec 5x \tan 5x \frac{d}{dx}(5x) + \sec^2 5x \frac{d}{dx}(5x)}{\sec 5x + \tan 5x} \\ &= \frac{5\sec 5x \tan 5x + 5\sec^2 5x}{\sec 5x + \tan 5x} \\ &= \frac{5\sec 5x(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} \\ &= 5\sec 5x \end{aligned}$$

(r) $y = 10^{\tan \pi\theta}$

Solution:

$$\begin{aligned} y' &= 10^{\tan\pi\theta}(\ln 10)\frac{d}{d\theta}(\tan\pi\theta) \\ &= 10^{\tan\pi\theta}(\ln 10)(\sec^2\pi\theta)\frac{d}{d\theta}(\pi\theta) \\ &= 10^{\tan\pi\theta}(\ln 10)(\sec^2\pi\theta)(\pi) \\ &= \pi 10^{\tan\pi\theta}(\ln 10)\sec^2\pi\theta \end{aligned}$$

(s) $y = \sqrt{t\ln(t^4)}$

Solution:

$$\begin{aligned}
y &= (t\ln(t^4))^{1/2} \\
y' &= \frac{1}{2}(t\ln(t^4))^{-1/2}\frac{d}{dt}(t\ln(t^4)) \\
&= \frac{1}{2}(t\ln(t^4))^{-1/2}\left(t\frac{d}{dt}(\ln(t^4)) + \ln(t^4)\frac{d}{dt}(t)\right) \\
&= \frac{1}{2}(t\ln(t^4))^{-1/2}\left(t\left(\frac{4t^3}{t^4}\right) + \ln(t^4)(1)\right) \\
&= \frac{1}{2\sqrt{t\ln(t^4)}}\left(4 + \ln(t^4)\right) \\
&= \frac{4 + \ln(t^4)}{2\sqrt{t\ln(t^4)}}
\end{aligned}$$

(t) $y = \arctan(\arcsin\sqrt{x})$

Solution:

$$\begin{aligned}
y &= \arctan(\arcsin x^{1/2}) \\
y' &= \frac{1}{1 + (\arcsin x^{1/2})^2}\ \frac{d}{dx}(\arcsin x^{1/2}) \\
&= \frac{1}{1 + (\arcsin x^{1/2})^2}\ \left(\frac{1}{\sqrt{1 - (x^{1/2})^2}}\ \frac{d}{dx}(x^{1/2})\right) \\
&= \frac{1}{1 + (\arcsin x^{1/2})^2}\ \left(\frac{1}{\sqrt{1 - x}}\left(\frac{1}{2}x^{-1/2}\right)\right) \\
&= \frac{1}{2(1 + (\arcsin\sqrt{x})^2)\sqrt{x(1 - x)}}
\end{aligned}$$

2. Find the slope of the line tangent to the curve at the specified point.

(a) $y = 4\sin^2 x$, $\left(\frac{\pi}{6}, 1\right)$

Solution:

$$\begin{aligned} y &= 4\sin^2 x \\ &= 4(\sin x)^2 \\ y' &= 4(2(\sin x)\frac{d}{dx}(\sin x)) \\ &= 8\sin x\cos x \end{aligned}$$

So the slope of the tangent line is

$$\begin{aligned} m &= 8\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) \\ &= 8\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= 2\sqrt{3} \end{aligned}$$

And the equation of the tangent line is

$$\begin{aligned} y - 1 &= 2\sqrt{3}\left(x - \frac{\pi}{6}\right) \\ y - 1 &= 2\sqrt{3}x - \frac{\pi\sqrt{3}}{3} \\ y &= 2\sqrt{3}x + 1 - \frac{\pi\sqrt{3}}{3} \end{aligned}$$

(b) $y = \dfrac{x^2-1}{x^2+1}$, $(0, -1)$

Solution:

$$\begin{aligned} y &= \frac{x^2-1}{x^2+1} \\ y' &= \frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} \\ &= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

So the slope of the tangent line is

$$\begin{aligned} m &= \frac{4(0)}{((0)^2+1)^2} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

And the equation of the tangent line is

$$\begin{aligned} y-(-1) &= (0)(x-0) \\ y+1 &= 0 \\ y &= -1 \end{aligned}$$

(c) $y = \sqrt{1 + 4\sin x}$, $(0, 1)$

Solution:

$$\begin{aligned} y &= \sqrt{1+4\sin x} \\ &= (1+4\sin x)^{1/2} \\ y' &= \frac{1}{2}(1+4\sin x)^{-1/2}\frac{d}{dx}(1+4\sin x) \\ &= \frac{1}{2}(1+4\sin x)^{-1/2}(4\cos x) \\ &= \frac{2\cos x}{\sqrt{1+4\sin x}} \end{aligned}$$

So the slope of the tangent line is

$$\begin{aligned} m &= \frac{2\cos(0)}{\sqrt{1+4\sin(0)}} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

And the equation of the tangent line is

$$\begin{aligned} y-1 &= 2(x-0) \\ y-1 &= 2x \\ y &= 2x+1 \end{aligned}$$

(d) $y = (2 + x)e^{-x}$, $(0, 2)$

Solution:

$$\begin{aligned} y &= (2+x)e^{-x} \\ y' &= (2+x)\frac{d}{dx}(e^{-x}) + (e^{-x})\frac{d}{dx}(2+x) \\ &= (2+x)e^{-x}\frac{d}{dx}(-x) + e^{-x}(1) \\ &= (2+x)e^{-x}(-1) + e^{-x} \\ &= (-2-x)e^{-x} + e^{-x} \\ &= e^{-x}(-2-x+1) \\ &= e^{-x}(-1-x) \end{aligned}$$

So the slope of the tangent line is

$$\begin{aligned} m &= e^{-(0)}(-1-(0)) \\ &= (1)(-1) \\ &= -1 \end{aligned}$$

And the equation of the tangent line is

$$\begin{aligned} y-2 &= (-1)(x-0) \\ y-2 &= -x \\ y &= -x+2 \end{aligned}$$

3. At what point on the curve $y = [\ln(x+4)]^2$ is the tangent horizontal?

Solution:

$$\begin{aligned} y &= [\ln(x+4)]^2 \\ y' &= 2[\ln(x+4)]\frac{d}{dx}[\ln(x+4)] \\ &= 2\ln(x+4)\left(\frac{\frac{d}{dx}(x+4)}{x+4}\right) \\ &= 2\ln(x+4)\left(\frac{1}{x+4}\right) \\ &= \frac{2\ln(x+4)}{x+4} \end{aligned}$$

To find where the tangent is horizontal, set $y' = 0$ and solve for x:

$$\begin{aligned} \frac{2\ln(x+4)}{x+4} &= 0 \\ 2\ln(x+4) &= 0 \\ \ln(x+4) &= 0 \\ e^{\ln(x+4)} &= e^0 \\ x+4 &= 1 \\ x &= -3 \end{aligned}$$

If $x = -3$, then

$$y = [\ln(-3+4)]^2 = [\ln(1)]^2 = 0^2 = 0$$

so the tangent is horizontal at the point $(-3, 0)$.

4. (a) Find an equation of the line tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$

Solution:

$$\begin{aligned} x - 4y &= 1 \\ -4y &= -x + 1 \\ y &= \frac{1}{4}x - \frac{1}{4} \end{aligned}$$

Since parallel lines have the same slope, we want a tangent line with slope $m = \dfrac{1}{4}$. But the slope of the tangent line is given by

$$y' = e^x$$

so we have

$$\begin{aligned} \frac{1}{4} &= e^x \\ \ln\frac{1}{4} &= \ln e^x \\ \ln 4^{-1} &= x \\ -\ln 4 &= x \end{aligned}$$

Since $y = e^x$, we have $y = e^{\ln 1/4} = \frac{1}{4}$, so the equation of our tangent line is

$$\begin{aligned} y - \frac{1}{4} &= \frac{1}{4}(x - (-\ln 4)) \\ y - \frac{1}{4} &= \frac{1}{4}x + \frac{1}{4}\ln 4 \\ y &= \frac{1}{4}x + \frac{1}{4}\ln 4 + \frac{1}{4} \end{aligned}$$

(b) Find an equation of the line tangent to the curve $y = e^x$ that passes through the origin.

Solution: Let the point of tangency be (x, e^x). If the tangent line goes through the origin, it goes through the point $(0, 0)$. Since we have two points on the tangent line we can compute the slope of the tangent line

$$\begin{aligned} m &= \frac{e^x - 0}{x - 0} \\ &= \frac{e^x}{x} \end{aligned}$$

But the slope of the tangent line is given by $y' = e^x$ so we have

$$\begin{aligned} \frac{e^x}{x} &= e^x \\ e^x &= xe^x \\ 0 &= xe^x - e^x \\ 0 &= e^x(x-1) \end{aligned}$$

So either $e^x = 0$ (which is impossible) or $x - 1 = 0$. Therefore $x = 1$. If $x = 1$, we have $m = e^{(1)}$ so the equation of the tangent line is

$$\begin{aligned} y - 0 &= e(x-0) \\ y &= ex \end{aligned}$$

5. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$, and $f'(5) = 11$. Find

(a) $h'(2)$

Solution:

$$\begin{aligned} h'(x) &= f(x)g'(x) + f'(x)g(x) \\ h'(2) &= f(2)g'(2) + f'(2)g(2) \\ &= (3)(4) + (-2)(5) \\ &= 2 \end{aligned}$$

(b) $F'(2)$

Solution:

$$\begin{aligned} F'(x) &= f'(g(x))g'(x) \\ F'(2) &= f'(g(2))g'(2) \\ &= f'(5)(4) \\ &= (11)(4) \\ &= 44 \end{aligned}$$

6. Find f' in terms of g'.

(a) $f(x) = x^2 g(x)$

Solution: Use the product rule:

$$\begin{aligned} f'(x) &= x^2 \frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[x^2] \\ &= x^2 g'(x) + 2xg(x) \end{aligned}$$

(b) $f(x) = g(x^2)$

Solution: Use the chain rule:

$$\begin{aligned} f'(x) &= g'(x^2)\frac{d}{dx}[x^2] \\ &= 2xg'(x^2) \end{aligned}$$

(c) $f(x) = [g(x)]^2$

Solution: Use the chain rule:

$$f'(x) = 2[g(x)]g'(x)$$

(d) $f(x) = g(g(x))$

Solution: Use the chain rule:

$$f'(x) = g'(g(x))g'(x)$$

(e) $f(x) = g(e^x)$

Solution: Use the chain rule:

$$\begin{aligned} f'(x) &= g'(e^x)\frac{d}{dx}[e^x] \\ &= g'(e^x)e^x \end{aligned}$$

(f) $f(x) = e^{g(x)}$

Solution: Use the chain rule:

$$f'(x) = e^{g(x)} g'(x)$$

(g) $f(x) = \ln |g(x)|$

Solution: Use the chain rule:

$$f'(x) = \frac{g'(x)}{g(x)}$$

(h) $f(x) = g(\ln x)$

Solution: Use the chain rule:

$$\begin{aligned} f'(x) &= g'(\ln x)\frac{d}{dx}[\ln x] \\ &= g'(\ln x)\left(\frac{1}{x}\right) \\ &= \frac{g'(\ln x)}{x} \end{aligned}$$

7. Find $\frac{dy}{dx}$ using implicit differentiation.

(a) $xy^4 + x^2y = x + 3y$

Solution:

$$\begin{aligned} \frac{d}{dx}(xy^4 + x^2y) &= \frac{d}{dx}(x + 3y) \\ x\frac{d}{dx}(y^4) + y^4\frac{d}{dx}(x) + x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) &= 1 + 3\frac{dy}{dx} \\ x\left(4y^3\frac{dy}{dx}\right) + y^4(1) + x^2\frac{dy}{dx} + y(2x) &= 1 + 3\frac{dy}{dx} \\ 4xy^3\frac{dy}{dx} + x^2\frac{dy}{dx} - 3\frac{dy}{dx} &= 1 - y^4 - 2xy \\ (4xy^3 + x^2 - 3)\frac{dy}{dx} &= 1 - y^4 - 2xy \\ \frac{dy}{dx} &= \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3} \end{aligned}$$

(b) $x^2 \cos y + \sin 2y = xy$

Solution:

$$\begin{aligned}
\frac{d}{dx}(x^2\cos y + \sin 2y) &= \frac{d}{dx}(xy) \\
x^2\frac{d}{dx}(\cos y) + \cos y\frac{d}{dx}(x^2) + \frac{d}{dx}(\sin 2y) &= x\frac{d}{dx}(y) + y\frac{d}{dx}(x) \\
x^2(-\sin y)\frac{dy}{dx} + \cos y(2x) + \cos 2y\frac{d}{dx}(2y) &= x\frac{dy}{dx} + y(1) \\
-x^2\sin y\frac{dy}{dx} + 2x\cos y + \cos 2y\left(2\frac{dy}{dx}\right) &= x\frac{dy}{dx} + y \\
-x^2\sin y\frac{dy}{dx} + 2\cos 2y\frac{dy}{dx} - x\frac{dy}{dx} &= y - 2x\cos y \\
(-x^2\sin y + 2\cos 2y - x)\frac{dy}{dx} &= y - 2x\cos y \\
\frac{dy}{dx} &= \frac{y - 2x\cos y}{-x^2\sin y + 2\cos 2y - x}
\end{aligned}$$

(c) $\sin(xy) = x^2 - y$

Solution:

$$\begin{aligned}
\frac{d}{dx}[\sin(xy)] &= \frac{d}{dx}(x^2 - y) \\
\cos(xy)\frac{d}{dx}(xy) &= 2x - \frac{dy}{dx} \\
\cos(xy)\left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right) &= 2x - \frac{dy}{dx} \\
\cos(xy)\left(x\frac{dy}{dx} + y(1)\right) &= 2x - \frac{dy}{dx} \\
x\cos(xy)\frac{dy}{dx} + y\cos(xy) &= 2x - \frac{dy}{dx} \\
x\cos(xy)\frac{dy}{dx} + \frac{dy}{dx} &= 2x - y\cos(xy) \\
\left(x\cos(xy) + 1\right)\frac{dy}{dx} &= 2x - y\cos(xy) \\
\frac{dy}{dx} &= \frac{2x - y\cos(xy)}{x\cos(xy) + 1}
\end{aligned}$$

(d) $xe^y = y - 1$

Solution:

$$\begin{aligned}
\frac{d}{dx}(xe^y) &= \frac{d}{dx}(y-1) \\
x\frac{d}{dx}(e^y) + e^y\frac{d}{dx}(x) &= \frac{dy}{dx} \\
xe^y\frac{dy}{dx} + e^y(1) &= \frac{dy}{dx} \\
e^y &= \frac{dy}{dx} - xe^y\frac{dy}{dx} \\
e^y &= (1 - xe^y)\frac{dy}{dx} \\
\frac{e^y}{1 - xe^y} &= \frac{dy}{dx}
\end{aligned}$$

8. Find the slope of the line tangent to $x^2 + 4xy + y^2 = 13$ at the point $(2, 1)$.

Solution: Use implicit differentiation

$$\begin{aligned}
\frac{d}{dx}(x^2 + 4xy + y^2) &= \frac{d}{dx}(13) \\
2x + 4x\frac{d}{dx}(y) + y\frac{d}{dx}(4x) + 2y\frac{dy}{dx} &= 0 \\
2x + 4x\frac{dy}{dx} + 4y + 2y\frac{dy}{dx} &= 0 \\
4x\frac{dy}{dx} + 2y\frac{dy}{dx} &= -2x - 4y \\
(4x + 2y)\frac{dy}{dx} &= -2x - 4y \\
\frac{dy}{dx} &= \frac{-2x - 4y}{4x + 2y} \\
&= \frac{-x - 2y}{2x + y}
\end{aligned}$$

So the slope of the tangent line is

$$\begin{aligned}
m &= \frac{-(2) - 2(1)}{2(2) + (1)} \\
&= \frac{-4}{5} \\
&= -\frac{4}{5}
\end{aligned}$$

9. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

Solution: Use implicit differentiation

$$\begin{aligned}
\frac{d}{dx}(x^2 + 2y^2) &= \frac{d}{dx}(1) \\
2x + 4y\frac{dy}{dx} &= 0 \\
4y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{x}{2y}
\end{aligned}$$

To find points at which the slope is 1, we set $\frac{dy}{dx} = 1$

$$\begin{aligned}
1 &= -\frac{x}{2y} \\
2y &= -x \\
-2y &= x
\end{aligned}$$

If we substitute this back into the original equation, we will have

$$\begin{aligned}
(-2y)^2 + 2y^2 &= 1 \\
4y^2 + 2y^2 &= 1 \\
6y^2 &= 1 \\
y^2 &= \frac{1}{6} \\
y &= \pm\frac{1}{\sqrt{6}}
\end{aligned}$$

If $y = \frac{1}{\sqrt{6}}$, then $x = -2\left(\frac{1}{\sqrt{6}}\right) = -\frac{2}{\sqrt{6}}$. If $y = -\frac{1}{\sqrt{6}}$, then $x = -2\left(-\frac{1}{\sqrt{6}}\right) = \frac{2}{\sqrt{6}}$.
So the points where the tangent line has slope 1 are

$$\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \qquad \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$

10. If $f(t) = \sqrt{4t+1}$ find $f''(2)$.

Solution:

$$\begin{aligned}
f(t) &= (4t+1)^{1/2} \\
f'(t) &= \frac{1}{2}(4t+1)^{-1/2}\frac{d}{dt}(4t+1) \\
&= \frac{1}{2}(4t+1)^{-1/2}(4) \\
&= 2(4t+1)^{-1/2} \\
f''(t) &= 2\left(-\frac{1}{2}(4t+1)^{-3/2}\frac{d}{dt}(4t+1)\right) \\
&= 2\left(-\frac{1}{2}(4t+1)^{-3/2}(4)\right) \\
&= -4(4t+1)^{-3/2} \\
&= -\frac{4}{\sqrt{(4t+1)^3}} \\
f''(2) &= -\frac{4}{\sqrt{(4(2)+1)^3}} \\
&= -\frac{4}{\sqrt{(9)^3}} \\
&= -\frac{4}{27}
\end{aligned}$$

11. If $g(\theta) = \theta \sin\theta$, find $g''\left(\frac{\pi}{6}\right)$

Solution:

$$\begin{aligned}
g'(\theta) &= \theta\frac{d}{d\theta}(\sin\theta) + (\sin\theta)\frac{d}{d\theta}(\theta) \\
&= \theta\cos\theta + \sin\theta \\
g''(\theta) &= \theta\frac{d}{d\theta}\cos\theta + (\cos\theta)\frac{d}{d\theta}(\theta) + \cos\theta \\
&= \theta(-\sin\theta) + \cos\theta + \cos\theta \\
&= -\theta\sin\theta + 2\cos\theta \\
g''\left(\frac{\pi}{6}\right) &= -\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{6}\right) \\
&= -\frac{\pi}{6}\left(\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) \\
&= -\frac{\pi}{12} + \sqrt{3}
\end{aligned}$$

12. Find y'' if $x^6 + y^6 = 1$.

Solution: Use implicit differentiation

$$\begin{aligned}
\frac{d}{dx}(x^6 + y^6) &= \frac{d}{dx}(1) \\
6x^5 + 6y^5\frac{dy}{dx} &= 0 \\
6y^5\frac{dy}{dx} &= -6x^5 \\
\frac{dy}{dx} &= -\frac{x^5}{y^5} \\
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(-\frac{x^5}{y^5}\right) \\
\frac{d^2y}{dx^2} &= \frac{y^5\frac{d}{dx}(-x^5) - (-x^5)\frac{d}{dx}(y^5)}{(y^5)^2} \\
&= \frac{y^5(-5x^4) + x^5\left(5y^4\frac{dy}{dx}\right)}{y^{10}} \\
&= \frac{-5x^4y^5 + 5x^5y^4\left(-\frac{x^5}{y^5}\right)}{y^{10}} \\
&= \frac{-5x^4y^5 - \frac{5x^{10}}{y}}{y^{10}} \\
&= \frac{-5x^4y^5 - \frac{5x^{10}}{y}}{y^{10}}\left(\frac{y}{y}\right) \\
&= \frac{-5x^4y^6 - 5x^{10}}{y^{11}} \\
&= \frac{-5x^4(y^6 + x^6)}{y^{11}} \\
&= \frac{-5x^4(1)}{y^{11}} \\
&= -\frac{5x^4}{y^{11}}
\end{aligned}$$

13. Find $f^{(n)}(x)$ if $f(x) = \dfrac{1}{2-x}$

Solution:

$$\begin{aligned}
f(x) &= (2-x)^{-1} \\
f'(x) &= -(2-x)^{-2}(-1) \\
&= (2-x)^{-2} \\
f''(x) &= -2(2-x)^{-3}(-1) \\
&= 2(2-x)^{-3} \\
f'''(x) &= 2(-3)(2-x)^{-4}(-1) \\
&= 2\cdot 3(2-x)^{-4} \\
f^{(4)}(x) &= 2\cdot 3(-4)(2-x)^{-5}(-1) \\
&= 2\cdot 3\cdot 4(2-x)^{-5} \\
f^{(5)}(x) &= 2\cdot 3\cdot 4(-5)(2-x)^{-6}(-1) \\
&= 2\cdot 3\cdot 4\cdot 5(2-x)^{-6} \\
f^{(n)}(x) &= 2\cdot 3\cdot 4\cdot 5\cdots n(2-x)^{-(n+1)} \\
&= n!(2-x)^{-n-1}
\end{aligned}$$

14. Calculate y' using logarithmic differentiation.

(a) $y = (\cos x)^x$

Solution:

$$\begin{aligned}
\ln y &= \ln(\cos x)^x \\
&= x\ln(\cos x) \\
\frac{y'}{y} &= x\frac{d}{dx}[\ln(\cos x)] + \ln(\cos x)\frac{d}{dx}[x] \\
\frac{y'}{y} &= x\left(\frac{\frac{d}{dx}(\cos x)}{\cos x}\right) + \ln(\cos x) \\
\frac{y'}{y} &= x\left(\frac{-\sin x}{\cos x}\right) + \ln(\cos x) \\
\frac{y'}{y} &= -x\tan x + \ln(\cos x) \\
y' &= y(-x\tan x + \ln(\cos x)) \\
&= (\cos x)^x(-x\tan x + \ln(\cos x))
\end{aligned}$$

(b) $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$

Solution:

$$\begin{aligned}
\ln y &= \ln\left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\right) \\
&= \ln(x^2+1)^4 - \ln(2x+1)^3 - \ln(3x-1)^5 \\
&= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \\
\frac{y'}{y} &= 4\left(\frac{\frac{d}{dx}(x^2+1)}{x^2+1}\right) - 3\left(\frac{\frac{d}{dx}(2x+1)}{2x+1}\right) - 5\left(\frac{\frac{d}{dx}(3x-1)}{3x-1}\right) \\
\frac{y'}{y} &= 4\left(\frac{2x}{x^2+1}\right) - 3\left(\frac{2}{2x+1}\right) - 5\left(\frac{3}{3x-1}\right) \\
\frac{y'}{y} &= \frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1} \\
y' &= y\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right) \\
&= \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right)
\end{aligned}$$

(c) $y = \dfrac{\sqrt{x+1}(2-x)^5}{(x+3)^7}$

Solution:

$$\begin{aligned}
\ln y &= \ln\left(\frac{(x+1)^{1/2}(2-x)^5}{(x+3)^7}\right) \\
&= \ln(x+1)^{1/2} + \ln(2-x)^5 - \ln(x+3)^7 \\
&= \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \\
\frac{y'}{y} &= \frac{1}{2}\left(\frac{\frac{d}{dx}(x+1)}{x+1}\right) + 5\left(\frac{\frac{d}{dx}(2-x)}{2-x}\right) - 7\left(\frac{\frac{d}{dx}(x+3)}{x+3}\right) \\
\frac{y'}{y} &= \frac{1}{2}\left(\frac{1}{x+1}\right) + 5\left(\frac{-1}{2-x}\right) - 7\left(\frac{1}{x+3}\right) \\
\frac{y'}{y} &= \frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3} \\
y' &= y\left(\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3}\right) \\
&= \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7}\left(\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+3}\right)
\end{aligned}$$

(d) $y = xe^{\sin x}$

Solution:

$$\begin{aligned}
\ln y &= \ln xe^{\sin x} \\
&= \ln x + \ln e^{\sin x} \\
&= \ln x + \sin x \\
\frac{y'}{y} &= \frac{1}{x} + \cos x \\
y' &= y\left(\frac{1}{x} + \cos x\right) \\
&= xe^{\sin x}\left(\frac{1}{x} + \cos x\right) \\
&= e^{\sin x}(1 + x\cos x)
\end{aligned}$$

15. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$, $t \geq 0$, where b and c are positive constants. Find the velocity and acceleration functions.

Solution:

$$\begin{aligned}
f(t) &= \sqrt{b^2 + c^2t^2} \\
&= (b^2 + c^2t^2)^{1/2} \\
v(t) &= f'(t) \\
&= \frac{1}{2}(b^2 + c^2t^2)^{-1/2}\frac{d}{dt}(b^2 + c^2t^2) \\
&= \frac{1}{2}(b^2 + c^2t^2)^{-1/2}(2c^2t) \\
&= c^2t(b^2 + c^2t^2)^{-1/2} \\
a(t) &= f''(t) \\
&= c^2t\frac{d}{dt}(b^2 + c^2t^2)^{-1/2} + (b^2 + c^2t^2)^{-1/2}\frac{d}{dt}(c^2t) \\
&= c^2t\left(-\frac{1}{2}(b^2 + c^2t^2)^{-3/2}\frac{d}{dt}(b^2 + c^2t^2)\right) + (b^2 + c^2t^2)^{-1/2}(c^2) \\
&= c^2t\left(-\frac{1}{2}(b^2 + c^2t^2)^{-3/2}(2c^2t)\right) + c^2(b^2 + c^2t^2)^{-1/2} \\
&= -c^4t^2(b^2 + c^2t^2)^{-3/2} + c^2(b^2 + c^2t^2)^{-1/2} \\
&= \frac{-c^4t^2}{(b^2 + c^2t^2)^{3/2}} + \frac{c^2}{(b^2 + c^2t^2)^{1/2}} \\
&= \frac{-c^4t^2 + c^2(b^2 + c^2t^2)}{(b^2 + c^2t^2)^{3/2}} \\
&= \frac{-c^4t^2 + c^2b^2 + c^4t^2}{(b^2 + c^2t^2)^{3/2}} \\
&= \frac{c^2b^2}{(b^2 + c^2t^2)^{3/2}}
\end{aligned}$$

16. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$. Find the velocity and acceleration functions.

Solution:

$$\begin{aligned} f(t) &= t^3 - 12t + 3 \\ v(t) &= f'(t) \\ &= 3t^2 - 12 \\ a(t) &= f''(t) \\ &= 6t \end{aligned}$$

17. The volume of a right circular cone is $V = \dfrac{\pi r^2 h}{3}$, where r is the radius of the base and h is the height.

(a) Find the rate of change of the volume with respect to the height if the radius is constant.

Solution:
$$\frac{dV}{dt} = \frac{\pi r^2}{3}\frac{dh}{dt}$$

(b) Find the rate of change of the volume with respect to the radius if the height is constant.

Solution:
$$\frac{dV}{dt} = \frac{\pi h(2r)}{3}\frac{dr}{dt} = \frac{2\pi rh}{3}\frac{dr}{dt}$$

18. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of the edge is 30 cm?

Solution: Let x=edge of the cube, V=volume of the cube, A=surface area of the cube. We are told that $\dfrac{dV}{dt} = 10\ \text{cm}^3/\text{min}$. We want to find $\dfrac{dA}{dt}$ when $x = 30$ cm. We also have the relationships

$$V = x^3 \qquad A = 6x^2$$

We can use these two equations to relate V and A. If we solve each equation for x, we have

$$x = V^{1/3} \qquad x = \frac{A^{1/2}}{6^{1/2}}$$

So we have the relationship

$$V^{1/3} = \frac{A^{1/2}}{6^{1/2}}$$

which can be written as

$$6^{1/2}V^{1/3} = A^{1/2}$$

So we have

$$\begin{aligned} \frac{1}{2}A^{-1/2}\frac{dA}{dt} &= 6^{1/2}\frac{1}{3}V^{-2/3}\frac{dV}{dt} \\ \frac{1}{2}A^{-1/2}\frac{dA}{dt} &= \frac{6^{1/2}}{3}V^{-2/3}\frac{dV}{dt} \end{aligned}$$

When $x = 30$ cm, we have $V = 30^3 = 27000$ cm^3 and $A = 6(30)^2 = 5400$ cm^2, so we have

$$\begin{aligned} \frac{1}{2}(5400)^{-1/2}\frac{dA}{dt} &= \frac{6^{1/2}}{3}(27000)^{-2/3}(10) \\ \frac{dA}{dt} &= \left(\frac{10 \cdot 6^{1/2}}{3(27000)^{2/3}}\right)\left(2(5400)^{1/2}\right) \\ &= \frac{4}{3}\text{cm}^2/\text{min} \end{aligned}$$

19. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

Solution: Let V=volume of water in the cup, h=height of water in the cup, and r=radius at the water level. So we know that $\frac{dV}{dt} = 2$ cm^3/s and we want to find $\frac{dh}{dt}$ when $h = 5$ cm. We can relate V, h and r by

$$V = \frac{1}{3}\pi r^2 h$$

but we need to relate V to h alone. We can find r in terms of h by using similar triangles

$$\frac{r}{3} = \frac{h}{10}$$

so $r = \frac{3h}{10}$ and our relationship is now

$$V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100}h^3$$

and so

$$\frac{dV}{dt} = \frac{3\pi}{100} 3h^2 \frac{dh}{dt} = \frac{9\pi h^2}{100} \frac{dh}{dt}$$

So we have

$$\begin{aligned} 2 &= \frac{9\pi(5)^2}{100} \frac{dh}{dt} \\ \frac{8}{9\pi} \text{ cm/s} &= \frac{dh}{dt} \end{aligned}$$

20. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

Solution: Let x=horizontal distance the boy has traveled since passing under the balloon, y=height of the balloon, and z=distance between the boy and the balloon. We know that

$$\frac{dx}{dt} = 15 \text{ ft/s} \qquad \frac{dy}{dt} = 5 \text{ ft/s}$$

and we want to know $\frac{dz}{dt}$ after 3 seconds, that is when $x = 15(3) = 45$ and $y = 45 + 5(3) = 60$ We know that

$$x^2 + y^2 = z^2$$

so we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

When $x = 45$ and $y = 60$ we have $z^2 = 45^2 + 60^2 = 5625$ so $z = 75$. That gives us

$$\begin{aligned} 2(45)(15) + 2(60)(5) &= 2(75)\frac{dz}{dt} \\ 1950 &= 150\frac{dz}{dt} \\ 13 \text{ ft/s} &= \frac{dz}{dt} \end{aligned}$$