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Math 121

Practice Exam #1
September 25, 2006

Name: ____________________
Section: ____________________

Directions: Answer the following questions on a separate piece of paper. You may not use a calculator. If you do not show your work, you will receive no credit. You may not use any short cut methods to calculate the derivative. **CIRCLE YOUR FINAL ANSWER. IF YOU DO NOT FOLLOW DIRECTIONS YOU WILL BE PENALIZED!** Note: This practice exam is much longer than the actual exam. It is meant to give an idea of types of questions that will be asked.

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1. Find $f \circ g$ and $g \circ f$ and their domains.

   (a) $f(x) = \ln x$, $g(x) = x^2 - 9$

Solution:

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f(x^2 - 9) \\
&= \ln(x^2 - 9) \\
\text{Dom}(f \circ g) &= \{x : x^2 - 9 > 0\} \\
&= \{x : x^2 > 9\} \\
&= \{x : |x| > 3\} \\
&= \{x : x > 3 \text{ or } x < -3\} \\
g \circ f &= g(f(x)) \\
&= g(\ln x) \\
&= (\ln x)^2 - 9 \\
\text{Dom}(g \circ f) &= \{x : x > 0\}
\end{aligned}$$

   (b) $f(x) = x^2$, $g(x) = \sqrt{x}$

Solution:

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f(\sqrt{x}) \\
&= (\sqrt{x})^2 \\
&= x \\
\text{Dom}(f \circ g) &= \{x : x > 0\} \\
g \circ f &= g(f(x)) \\
&= g(x^2) \\
&= \sqrt{x^2} \\
&= |x| \\
\text{Dom}(g \circ f) &= \mathbb{R}
\end{aligned}$$

(c) $f(x) = \frac{3}{x}$, $g(x) = x^2 - 1$

Solution:

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f(x^2 - 1) \\
&= \frac{3}{x^2 - 1} \\
\mathrm{Dom}(f \circ g) &= \{x : x^2 - 1 \neq 0\} \\
&= \{x : x^2 \neq 1\} \\
&= \{x : |x| \neq 1\} \\
&= \{x : x \neq 1 \text{ and } x \neq -1\} \\
g \circ f &= g(f(x)) \\
&= g\left(\frac{3}{x}\right) \\
&= \left(\frac{3}{x}\right)^2 - 1 \\
&= \frac{9}{x^2} - 1 \\
&= \frac{9 - x^2}{x^2} \\
\mathrm{Dom}(g \circ f) &= \{x : x \neq 0\}
\end{aligned}$$

2. Find the exact value of

(a) $\log_2 \frac{1}{8}$

Solution:

$$\begin{aligned}
\log_2 \frac{1}{8} &= \log_2 2^{-3} \\
&= -3
\end{aligned}$$

(b) $\log_{27} 9$

Solution:

$$\begin{aligned}
\log_{27} 9 &= \log_{27} 27^{2/3} \\
&= \frac{2}{3}
\end{aligned}$$

(c) $\log_5 1$

Solution:

$$\log_5 1 = 0$$

since $\log_b 1 = 0$ for all $b > 0$.

3. Write the expression as a single logarithm.

(a) $3\log_2 x + 2\log_2 y - 4\log_2 z$

Solution:

$$\begin{aligned} 3\log_2 x + 2\log_2 y - 4\log_2 z &= \log_2 x^3 + \log_2 y^2 - \log_2 z^4 \\ &= \log_2\left(\frac{x^3y^2}{z^4}\right) \end{aligned}$$

(b) $2\Big(\log_7 x - \log_7(x+1) - \log_7(x-1)\Big)$

Solution:

$$\begin{aligned} 2\Big(\log_7 x - \log_7(x+1) - \log_7(x-1)\Big) &= 2\log_7\left(\frac{x}{(x+1)(x-1)}\right) \\ &= \log_7\left(\frac{x}{x^2-1}\right)^2 \\ &= \log_7\left(\frac{x^2}{(x^2-1)^2}\right) \\ &= \log_7\left(\frac{x^2}{x^4-2x^2+1}\right) \end{aligned}$$

(c) $2\ln 3 - \frac{1}{2}\ln(x^2+1)$

Solution:

$$\begin{aligned} 2\ln 3 - \frac{1}{2}\ln(x^2+1) &= \ln 3^2 - \ln(x^2+1)^{1/2} \\ &= \ln\left(\frac{9}{\sqrt{x^2+1}}\right) \end{aligned}$$

4. Solve for x.

(a) $3^{2x} = 75$

Solution:

$$\begin{aligned} 3^{2x} &= 75 \\ \log_3 3^{2x} &= \log_3 75 \\ 2x &= \log_3 75 \\ x &= \frac{1}{2}\log_3 75 \\ x &= \log_3 75^{1/2} \\ x &= \log_3 5\sqrt{3} \end{aligned}$$

(b) $2^{3-x} = 625$

Solution:

$$\begin{aligned} 2^{3-x} &= 625 \\ \log_2 2^{3-x} &= \log_2 625 \\ 3 - x &= \log_2 625 \\ 3 - \log_2 625 &= x \end{aligned}$$

(c) $\log_2(x-1) = 5$

Solution:

$$\begin{aligned} \log_2(x-1) &= 5 \\ 2^{\log_2(x-1)} &= 2^5 \\ x - 1 &= 32 \\ x &= 33 \end{aligned}$$

(d) $\log_3 \sqrt{x-4} = 2$

Solution:

$$\begin{aligned} \log_3 \sqrt{x-4} &= 2 \\ 3^{\log_3 \sqrt{x-4}} &= 3^2 \\ \sqrt{x-4} &= 9 \\ x-4 &= 81 \\ x &= 85 \end{aligned}$$

5. Evaluate the following limits.

(a) $\lim_{x \to 3} \dfrac{x^2-9}{x^2+2x-3}$

Solution:

$$\begin{aligned} \lim_{x \to 3} \frac{x^2-9}{x^2+2x-3} &= \frac{(3)^2-9}{(3)^2+2(3)-3} \\ &= \frac{0}{12} \\ &= 0 \end{aligned}$$

(b) $\lim_{x \to 1^+} \dfrac{x^2-9}{x^2+2x-3}$

Solution: Direct substitution yields

$$\begin{aligned} \lim_{x \to 1^+} \frac{x^2-9}{x^2+2x-3} &= \frac{(1)^2-9}{(1)^2+2(1)-3} \\ &= \frac{-8}{0} \end{aligned}$$

Meaning that $\lim_{x \to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$

(c) $\lim_{x\to 4^+} \frac{4-x}{|4-x|}$

Solution: Direct substitution yields $\frac{0}{0}$. Since $x \to 4^+$, we have $x > 4$ which means $4 - x < 0$. So when $x \to 4^+$, we have $|4-x| = -(4-x)$.

$$\begin{aligned}
\lim_{x\to 4^+} \frac{4-x}{|4-x|} &= \lim_{x\to 4^+} \frac{4-x}{-(4-x)} \\
&= \lim_{x\to 4^+} (-1) \\
&= -1
\end{aligned}$$

(d) $\lim_{x\to 2} \frac{x^2+2x-8}{x^4-16}$

Solution: Direct substitution yields $\frac{0}{0}$, so we will factor

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2+2x-8}{x^4-16} &= \lim_{x\to 2} \frac{(x+4)(x-2)}{(x^2-4)(x^2+4)} \\
&= \lim_{x\to 2} \frac{(x+4)(x-2)}{(x-2)(x+2)(x^2+4)} \\
&= \lim_{x\to 2} \frac{x+4}{(x+2)(x^2+4)} \\
&= \frac{(2)+4}{((2)+2)((2)^2+4)} \\
&= \frac{6}{4(8)} \\
&= \frac{3}{16}
\end{aligned}$$

(e) $\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x}$

Solution: Direct substitution yields $\frac{0}{0}$, so we will multiply by the conjugate

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x} &= \lim_{x\to 2} \left(\frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x}\right)\left(\frac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}}\right) \\
&= \lim_{x\to 2} \frac{(x+2)-(2x)}{(x^2-2x)(\sqrt{x+2}+\sqrt{2x})} \\
&= \lim_{x\to 2} \frac{-x+2}{-x(-x+2)(\sqrt{x+2}+\sqrt{2x})} \\
&= \lim_{x\to 2} \frac{1}{-x(\sqrt{x+2}+\sqrt{2x})} \\
&= \frac{1}{-(2)(\sqrt{(2)+2}+\sqrt{2(2)})} \\
&= \frac{1}{-2(2+2)} \\
&= -\frac{1}{8}
\end{aligned}$$

(f) $\lim_{x\to -\infty} \frac{5x^3-x^2+2}{2x^3+x-3}$

Solution:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{5x^3-x^2+2}{2x^3+x-3} &= \lim_{x\to -\infty} \frac{\frac{5x^3-x^2+2}{x^3}}{\frac{2x^3+x-3}{x^3}} \\
&= \lim_{x\to -\infty} \frac{5-\frac{1}{x}+\frac{2}{x^3}}{2+\frac{1}{x^2}-\frac{3}{x^3}} \\
&= \frac{5-0+0}{2+0-0} \\
&= \frac{5}{2}
\end{aligned}$$

(g) $\lim_{x\to 10^-} \ln(100-x^2)$

Solution: We can not use direct substitution because we cannot take the log of zero. However, we can reason that if $x \to 10^-$, then x^2 will be close to, but smaller than 100. So $100-x^2$ will be close to 0 and positive. The natural log of a small positive number is large and negative (think of the graph of $y = \ln x$).

So we say $\lim_{x\to 10^-} \ln(100 - x^2) = -\infty$

6. Let

$$f(x) = \begin{cases} \sqrt{-x} & x < 0 \\ 3 - x & 0 \le x < 3 \\ (x-3)^2 & x > 3 \end{cases}$$

(a) Evaluate the following limits

i. $\lim_{x\to 0^+} f(x)$

Solution: When $x \to 0^+$, we have $x > 0$ (but close to 0). So we use the second rule for the function:

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (3 - x) \\ &= 3 - (0) \\ &= 3 \end{aligned}$$

ii. $\lim_{x\to 0^-} f(x)$

Solution: When $x \to 0^-$, we have $x < 0$ (but close to 0). So we use the first rule for the function:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \sqrt{-x} \\ &= \sqrt{-(0)} \\ &= 0 \end{aligned}$$

iii. $\lim_{x\to 0} f(x)$

Solution: Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, the limit $\lim_{x\to 0} f(x)$ does not exist.

iv. $\lim_{x\to 3^+} f(x)$

Solution: When $x \to 3^+$, we have $x > 3$ (but close to 3). So we use the third rule for the function:

$$\begin{aligned} \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (x-3)^2 \\ &= ((3) - 3)^2 \\ &= 0 \end{aligned}$$

v. $\lim_{x \to 3^-} f(x)$

Solution: When $x \to 3^-$, we have $x < 3$ (but close to 3). So we use the second rule for the function:

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} (3 - x) \\ &= 3 - (3) \\ &= 0 \end{aligned}$$

vi. $\lim_{x \to 3} f(x)$

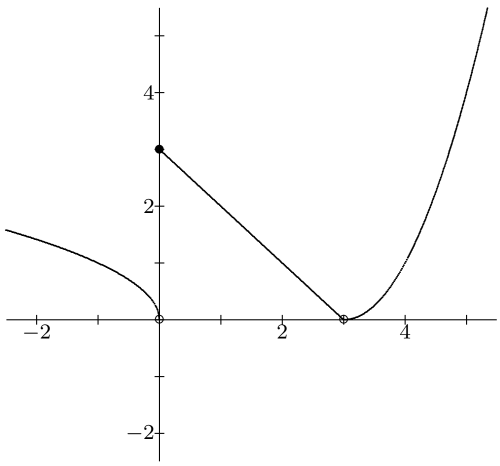
Solution: Since $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = 0$, se have $\lim_{x \to 3} f(x) = 0$.

(b) Where is $f(x)$ discontinuous?

Solution: Each "piece" of the function is continuous on its domain, so we only have to worry about where the pieces might potentially meet, i.e. $x = 0$ and $x = 3$. Since $\lim_{x \to 0} f(x)$ does not exist, the function is not continuous at $x = 0$. Notice that $f(x)$ is not defined at $x = 3$, so the function is not continuous at $x = 3$. Thus the points of discontinuity are when $x = 0, 3$.

(c) Sketch a graph of f.

Solution:



7. Let

$$g(x) = \begin{cases} 2x - x^2 & 0 \le x \le 2 \\ 2 - x & 2 < x \le 3 \\ x - 4 & 3 < x < 4 \\ \pi & x \ge 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, determine if g is continuous from the right, continuous from the left, or continuous at the number.

Solution: For $x = 2$, we have

$$\begin{aligned} \lim_{x\to 2^-} g(x) &= \lim_{x\to 2^-} (2x - x^2) \\ &= 2(2) - (2)^2 \\ &= 0 \\ \lim_{x\to 2^+} g(x) &= \lim_{x\to 2^+} (2 - x) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

So $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = 0$, which means $\lim_{x\to 2} g(x) = 0$. Notice that $g(2) = 0$, so g is continuous at $x = 2$.
For $x = 3$, we have

$$\begin{aligned} \lim_{x\to 3^-} g(x) &= \lim_{x\to 3^-} (2 - x) \\ &= 2 - (3) \\ &= -1 \\ \lim_{x\to 3^+} g(x) &= \lim_{x\to 3^+} (x - 4) \\ &= (3) - 4 \\ &= -1 \end{aligned}$$

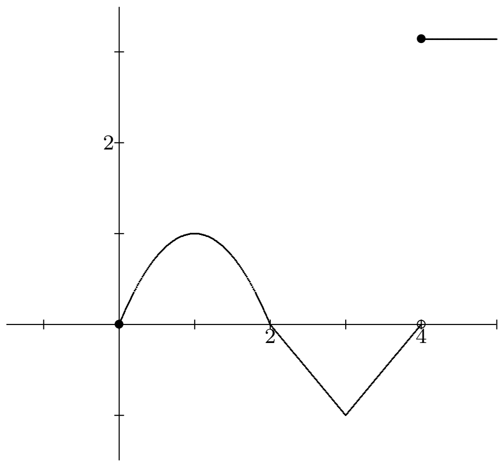
So $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^+} g(x) = -1$, which means $\lim_{x\to 3} g(x) = -1$. Notice that $g(3) = -1$, so g is continuous at $x = 3$.
For $x = 4$, we have

$$\begin{aligned} \lim_{x\to 4^-} g(x) &= \lim_{x\to 4^-} (x - 4) \\ &= (4) - 4 \\ &= 0 \\ \lim_{x\to 4^+} g(x) &= \lim_{x\to 4^+} \pi \\ &= \pi \end{aligned}$$

So $\lim_{x\to 4^-} g(x) \neq \lim_{x\to 4^+} g(x)$, so g is not continuous at $x = 4$. However since $g(4) = \pi = \lim_{x\to 4^+} g(x)$, g is continuous from the right at $x = 4$.

(b) Sketch a graph of g.

Solution:



8. Find the slope of the line tangent to $y = 9 - 2x^2$ at the point $(2, 1)$. Use that to write the equation of the tangent line.

Solution: The slope of the tangent line to $f(x) = 9 - 2x^2$ when $x = 2$ is

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{(9 - 2(2+h)^2) - (9 - 2(2)^2)}{h} \\
&= \lim_{h\to 0} \frac{9 - 2(4 + 4h + h^2) - 1}{h} \\
&= \lim_{h\to 0} \frac{9 - 8 - 8h - 2h^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{-8h - 2h^2}{h} \\
&= \lim_{h\to 0} \frac{h(-8 - 2h)}{h} \\
&= \lim_{h\to 0} (-8 - 2h) \\
&= -8 - (0) \\
&= -8
\end{aligned}$$

So we want the line with slope $f'(2) = -8$ through the point $(2, 1)$, which is

$$\begin{aligned} y - 1 &= -8(x-2) \\ y - 1 &= -8x + 16 \\ y &= -8x + 17 \end{aligned}$$

9. Find the equations of the tangent lines to the curve

$$y = \frac{2}{1-3x}$$

at the points where the x-coordinate is 0 and -1.

Solution: We need the derivative at two values of x, so I will compute a formula for $f'(x)$ first:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{1-3(x+h)} - \frac{2}{1-3x}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{2(1-3x) - 2(1-3x-3h)}{(1-3x-3h)(1-3x)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{2-6x-2+6x+6h}{(1-3x-3h)(1-3x)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{6h}{(1-3x-3h)(1-3x)}\right) \\ &= \lim_{h\to 0} \left(\frac{6}{(1-3x-3h)(1-3x)}\right) \\ &= \frac{6}{(1-3x-3(0))(1-3x)} \\ &= \frac{6}{(1-3x)^2} \end{aligned}$$

When our x-coordinate is 0, we have $f(0) = \frac{2}{1-3(0)} = 2$ and $f'(0) = \frac{6}{(1-3(0))^2} = 6$. Thus when $x = 0$ we are at the point $(0, 2)$ and the tangent line has slope $f'(0) = 6$. So the equation of the tangent line at $x = 0$ is

$$\begin{aligned} y - 2 &= 6(x-0) \\ y - 2 &= 6x \\ y &= 6x + 2 \end{aligned}$$

When our x-coordinate is 1, we have $f(1) = \frac{2}{1-3(1)} = -1$ and $f'(1) = \frac{6}{(1-3(1))^2} = \frac{3}{2}$. Thus when $x = 1$ we are at the point $(1, -1)$ and the tangent line has slope $f'(1) = \frac{3}{2}$. So the equation of the tangent line at $x = 1$ is

$$\begin{aligned} y - (-1) &= \frac{3}{2}(x-1) \\ y + 1 &= \frac{3}{2}x - \frac{3}{2} \\ y &= \frac{3}{2}x - \frac{5}{2} \end{aligned}$$

10. Find the derivative of the following

(a) $f(x) = x^3 - 2x$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^3 - 2(x+h)] - [x^3 - 2x]}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h - x^3 + 2x}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 2)}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 2) \\ &= 3x^2 + 3x(0) + (0)^2 - 2 \\ &= 3x^2 - 2 \end{aligned}$$

(b) $f(x) = \sqrt{3 - 5x}$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{3 - 5(x+h)} - \sqrt{3 - 5x}}{h} \\
&= \lim_{h \to 0} \left(\frac{\sqrt{3 - 5x - 5h} - \sqrt{3 - 5x}}{h} \right) \left(\frac{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}}{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}} \right) \\
&= \lim_{h \to 0} \frac{(3 - 5x - 5h) - (3 - 5x)}{h(\sqrt{3 - 5x - 5h}) + \sqrt{3 - 5x})} \\
&= \lim_{h \to 0} \frac{-5h}{h(\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x})} \\
&= \lim_{h \to 0} \frac{-5}{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}} \\
&= \frac{-5}{\sqrt{3 - 5x - 5(0)} + \sqrt{3 - 5x}} \\
&= -\frac{5}{2\sqrt{3 - 5x}}
\end{aligned}$$

(c) $f(x) = \frac{4-x}{3+x}$

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{4-(x+h)}{3+(x+h)} - \frac{4-x}{3+x}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{(4 - x - h)(3 + x) - (4 - x)(3 + x + h)}{(3 + x + h)(3 + x)} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{(12 + 4x - 3x - x^2 - 3h - xh) - (12 + 4x + 4h - 3x - x^2 - xh)}{(3 + x + h)(3 + x)} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{-7h}{(3 + x + h)(3 + x)} \right) \\
&= \lim_{h \to 0} \left(\frac{-7}{(3 + x + h)(3 + x)} \right) \\
&= \frac{-7}{(3 + x + (0))(3 + x)} \\
&= -\frac{7}{(3 + x)^2}
\end{aligned}$$