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A Topological Proof and Extension of the Leggett-Williams Fixed Point Theorem

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Abstract

In this paper the Leggett-Williams fixed point theorem is extended by not requiring either of the functional boundaries in the arguments to be invariant. The justification for the functional fixed point theorems currently in the literature apply index theory, whereas this is the first that applies topological methods.

Key words: Fixed-point theorems, Leggett-Williams, topological methods.

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1 Introduction

In this paper we present a topological proof and extension of the Leggett-Williams fixed point theorem. Axiomatic index theory has been used to create and extend many functional fixed point theorems. Functional fixed point theorems (including [2, 3, 4, 5, 7]) can be traced back to Leggett and Williams [6] when they presented criteria which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regards to the concave functional boundary of a functional wedge. Avery, Henderson, and O'Regan [1], in a dual of the Leggett-Williams fixed point theorem, gave conditions which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regards to the convex functional boundary of a functional wedge. In this paper we apply topological techniques which enable us to guarantee the existence of a fixed point for a completely continuous map that does not require either of the functional boundaries to be invariant with respect to the functional wedge. Moreover, the justification is rather simple compared to the Leggett-Williams fixed point theorem since axiomatic index theory is not employed.

2 Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 2.1 *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a* cone *if it satisfies the following two conditions:*

(i) $x \in P, \lambda \geq 0$ *implies* $\lambda x \in P$*;*

(ii) $x \in P, -x \in P$ *implies* $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2 *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Definition 2.3 *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if*

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let α and ψ be non-negative continuous concave functionals on P and δ and β be non-negative continuous convex functionals on P; then, for non-negative real numbers a, b, c and d, we define the following sets:

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \tag{2.1}$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leq b\} \tag{2.2}$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leq \psi(x)\}. \tag{2.3}$$

We say that A is a *functional wedge with concave functional boundary* defined by the concave functional α and convex functional boundary defined by the convex functional β. We say that an operator $T : A \to P$ is *invariant with respect to the concave functional boundary*, if $a \leq \alpha(Tx)$ for all $x \in A$, and that T is *invariant with respect to the convex functional boundary*, if $\beta(Tx) \leq d$ for all $x \in A$. Note that A is a convex set. We will invoke the Schauder Fixed Point Theorem in our main arguments. The following statement of the Schauder Fixed Point Theorem can be found in Zeidler [8].

Theorem 2.4 *Let M be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose that $T : M \to M$ is a completely continuous operator. Then T has a fixed point in M.*

3 Main Result

Theorem 3.1 *Suppose P is a cone in a real Banach space E, α and ψ are non-negative continuous concave functionals on P, δ and β are non-negative continuous convex functionals on P, and for non-negative real numbers a, b, c and d the sets A, B and C are as defined in* (2.1), (2.2) *and* (2.3). *Furthermore, suppose that A is a bounded subset of P, that $T : A \to P$ is completely continuous and that the following conditions hold:*

(A1) $\{x \in A \,:\, c < \psi(x)$ *and* $\delta(x) < b\} \neq \emptyset$ *and* $\{x \in P \,:\, \alpha(x) < a$ *and* $d < \beta(x)\} = \emptyset$*;*

(A2) $\alpha(Tx) \geq a$ *for all* $x \in B$*;*

(A3) $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $\delta(Tx) > b$*;*

(A4) $\beta(Tx) \leq d$ *for all* $x \in C$*; and,*

(A5) $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $\psi(Tx) < c$*.*

Then T has a fixed point $x^* \in A$.

Proof: Let

$$x_0 \in \{x \in A \,:\, c < \psi(x) \text{ and } \delta(x) < b\}$$

and define the function $t : A \to [0,1]$ by

$$t_x := t(x) = \begin{cases} \frac{\alpha(x_0)-a}{\alpha(x_0)-\alpha(Tx)}, & \alpha(Tx) < a, \\ 1, & a \le \alpha(Tx) \text{ and } \beta(Tx) \le d, \\ \frac{d-\beta(x_0)}{\beta(Tx)-\beta(x_0)}, & d < \beta(Tx), \end{cases}$$

for each $x \in A$. Since α and β are continuous and

$$\{x \in P \ : \ \alpha(x) < a \text{ and } b < \beta(x)\} = \emptyset,$$

we have that t is well defined and continuous. Define the operator $S : A \to P$ by

$$Sx = t_x Tx + (1 - t_x)x_0,$$

for each $x \in A$.

Claim 1: $S : A \to A$.

Let $x \in A$. We proceed in cases, depending on the value of t_x.

Case 1: $\alpha(Tx) < a$.

Thus $t_x = \frac{\alpha(x_0)-a}{\alpha(x_0)-\alpha(Tx)}$ which implies that

$$\begin{aligned} \alpha(Sx) &= \alpha(t_x Tx + (1-t_x)x_0) \\ &\ge t_x\alpha(Tx) + (1-t_x)\alpha(x_0) \\ &= \left(\frac{\alpha(x_0)-a}{\alpha(x_0)-\alpha(Tx)}\right)\alpha(Tx) + \left(1 - \frac{\alpha(x_0)-a}{\alpha(x_0)-\alpha(Tx)}\right)\alpha(x_0) \\ &= a, \end{aligned}$$

and since $\{x \in P \ : \ \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$, we have $\beta(Tx) \le d$. This implies that

$$\begin{aligned} \beta(Sx) &= \beta(t_x Tx + (1-t_x)x_0) \\ &\le t_x\beta(Tx) + (1-t_x)\beta(x_0) \\ &\le d, \end{aligned}$$

and hence $Sx \in A$.

Case 2: $a \le \alpha(Tx)$ and $\beta(Tx) \le d$.

Thus $t_x = 1$ which implies that $Sx = Tx$. Hence $a \le \alpha(Tx) = \alpha(Sx)$ and $\beta(Sx) = \beta(Tx) \le d$, and therefore $Sx \in A$.

Case 3: $d < \beta(Tx)$.

Thus $t_x = \frac{d-\beta(x_0)}{\beta(Tx)-\beta(x_0)}$ which implies that

$$\begin{aligned}\beta(Sx) &= \beta(t_x Tx + (1-t_x)x_0) \\ &\leq \left(\frac{d-\beta(x_0)}{\beta(Tx)-\beta(x_0)}\right)\beta(Tx) + \left(1 - \frac{d-\beta(x_0)}{\beta(Tx)-\beta(x_0)}\right)\beta(x_0) \\ &= d,\end{aligned}$$

and since $\{x \in P \,:\, \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$, we have $\alpha(Tx) \geq a$. This implies that

$$\begin{aligned}\alpha(Sx) &= \alpha(t_x Tx + (1-t_x)x_0) \\ &\geq t_x\alpha(Tx) + (1-t_x)\alpha(x_0) \\ &\geq a,\end{aligned}$$

and hence $Sx \in A$.

Therefore, regardless of the case, we have shown that $S : A \to A$.

Clearly S is continuous and if (x_n) is a sequence in A, then there is a subsequence (y_n) of (x_n) such that $t_{y_n} \to t^* \in [0,1]$, and since T is completely continuous, there is a subsequence (z_n) of (y_n) such that $Tz_n \to z^*$. Hence $Sz_n \to t^*z^* + (1-t^*)x_0$ and thus $S : A \to A$ is completely continuous. Therefore, by the Schauder Fixed Point Theorem, S has a fixed point $x^* \in A$.

Claim 2: $x^* = Tx^*$.

Either $\delta(Tx^*) < b$ or $\delta(Tx^*) \geq b$. Again, we proceed in cases.

Case 1: $\delta(Tx^*) \leq b$.

Then,

$$\begin{aligned}\delta(x^*) &= \delta(t_{x^*}Tx^* + (1-t_{x^*})x_0) \\ &\leq t_{x^*}\delta(Tx^*) + (1-t_{x^*})\delta(x_0) \\ &\leq b,\end{aligned}$$

and hence $x^* \in B$. Thus by $(A2)$, $\alpha(Tx^*) \geq a$.

Case 2: $\delta(Tx^*) > b$.

Thus by $(A3)$, $\alpha(Tx^*) \geq a$.

Therefore, in either case, we have shown that $\alpha(Tx^*) \geq a$. Next, either $\psi(Tx^*) < c$ or $\psi(Tx^*) \geq c$, and we again proceed in cases.

Case 1: $\psi(Tx^*) < c$.

Thus by $(A5)$, $\beta(Tx^*) \leq d$.

Case 2: $\psi(Tx^*) \geq c$.

Then,

$$\begin{aligned}\psi(x^*) &= \psi(t_{x^*}Tx^* + (1-t_{x^*})x_0)\\ &\geq t_{x^*}\psi(Tx^*) + (1-t_{x^*})\psi(x_0)\\ &\geq c,\end{aligned}$$

and hence $x^* \in C$. Thus by $(A4)$, $\beta(Tx^*) \leq d$.

Therefore, in either case, we have shown that $\beta(Tx^*) \leq d$. Hence, $t_{x^*} = 1$, which implies that $Sx^* = Tx^*$ and we have that x^* is a fixed point of T. □

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