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# NONLOCAL INITIAL VALUE PROBLEM FOR FIRST-ORDER DYNAMIC EQUATIONS ON TIME SCALES

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**Abstract.** In this study, conditions for the existence of at least one solution to a nonlinear first-order nonlocal initial value problem on time scales are discussed. The results extend previous work in the continuous case to the discrete, quantum, and general time scales setting, and are based on the Leray-Schauder fixed point theorem.
**Keywords.** Time scales; Nonlinear dynamic equations; Leray-Schauder fixed point theorem; Initial value problems; Existence.
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## 1 Introduction

We are interested in the first-order nonlocal time-scale initial value problem

$$\begin{cases} x^{\Delta}(t) = f\big(t, x^{\sigma}(t)\big), & t \in (a,b)_{\mathbb{T}}, \\ x(a) + \displaystyle\sum_{j=1}^{m} \gamma_j x(t_j) = 0 \end{cases} \tag{1.1}$$

where $m \geq 1$ and the points $t_j \in \mathbb{T}^{\kappa}$ for $j \in \{1, 2, \ldots, m\}$ with $a \leq t_1 \leq \cdots \leq t_m < b$;

$$1 + \sum_{j=1}^{m} \gamma_j \neq 0, \quad \gamma_j \in \mathbb{R},\ j \in \{1, \ldots, m\}; \tag{1.2}$$

the function $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(t,x)| \leq \begin{cases} w\big(t, |x|\big) & : t \in [a, t_m]_{\mathbb{T}} \\ p(t)q\big(|x|\big) & : t \in [t_m, b]_{\mathbb{T}}, \end{cases} \tag{1.3}$$

where $w : [a, t_m]_{\mathbb{T}} \times \mathbb{R}_+ \to \mathbb{R}_+$ is integrable and nondecreasing in its second argument; $p : [t_m, b]_{\mathbb{T}} \to \mathbb{R}_+$ is right-dense continuous; and $q : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing with $1/q$ integrable on $\mathbb{R}_+$. Moreover, we assume that there exists $R_0 > 0$ such that

$$\eta > R_0 \quad \text{implies} \quad \frac{1}{\eta} \int_a^{t_m} w(s, \eta)\Delta s < \frac{1}{A} \tag{1.4}$$

and

$$\int_{t_m}^{b} p(s)\Delta s < \int_{R_0^*}^{\infty} \frac{d\eta}{q(\eta)}, \tag{1.5}$$

where $\alpha := \left(1 + \sum_{j=1}^{m} \gamma_j\right)^{-1}$, $A := 1 + |\alpha| \sum_{j=1}^{m} |\gamma_j|$, and $R_0^* := A \int_a^{t_m} w(s, R_0)\Delta s$.

Problem (1.1) extends to general time scales the special case $\mathbb{T} = \mathbb{R}$; see Boucherif and Precup [5]. There has of late been interest in first-order problems on time scales. Anderson [1], Cabada and Vivero [6], Dai and Tisdell [7], Otero-Espinar and Vivero [9], Sun [11], Sun and Li [12], and Tian and Ge [13] all recently consider first-order boundary value problems on time scales, but none of them consider the nonlocal problem. For more general information concerning dynamic equations on time scales, introduced by Aulbach and Hilger [2] and Hilger [8], see the excellent text by Bohner and Peterson [3].

It is straightforward to check that problem (1.1) is equivalent to the following integral equation in $\mathrm{C}[a,b]_{\mathbb{T}}$

$$x(t) = \int_a^t f\big(s, x^{\sigma}(s)\big)\Delta s - \alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^{\sigma}(s)\big)\Delta s.$$

This can be viewed as a fixed point problem in $\mathrm{C}[a,b]_{\mathbb{T}}$ for the completely continuous operator $L : \mathrm{C}[a,b]_{\mathbb{T}} \to \mathrm{C}[a,b]_{\mathbb{T}}$ given by

$$Lx(t) = \int_a^t f\big(s, x^{\sigma}(s)\big)\Delta s - \alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^{\sigma}(s)\big)\Delta s.$$

Notice that $L$ appears as a sum of two integral operators, one, say of Fredholm type, whose values depend only on the restrictions of functions to $[a, t_m]_{\mathbb{T}}$,

$$L_F x(t) = \begin{cases} -\alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^\sigma(s)\big)\Delta s \\ \quad + \int_a^t f\big(s, x^\sigma(s)\big)\Delta s & : t < t_m, \\ -\alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^\sigma(s)\big)\Delta s \\ \quad + \int_a^{t_m} f\big(s, x^\sigma(s)\big)\Delta s & : t \geq t_m; \end{cases}$$

and the other, a Voltera type operator,

$$L_V x(t) = \begin{cases} 0 & : t < t_m, \\ \displaystyle\int_{t_m}^t f\big(s, x^\sigma(s)\big)\Delta s & : t \geq t_m, \end{cases}$$

depending on the restrictions of functions to $[t_m, b]_{\mathbb{T}}$. Thus the growth conditions on the nonlinearity $f$ in the sequel will be split into two parts, one for the subinterval containing the points involved by the nonlocal condition, and the other for the rest of the domain of definition.

To emphasize the generality and flexibility of the time scale setting, we note the following corollary equations. If $\mathbb{T} = \mathbb{Z}$, then for $t \in \{a+1, a+2, \cdots, b-1\}$, the nonlocal initial value problem (1.1) takes the form

$$\begin{cases} x(t+1) = x(t) + f\big(t, x(t+1)\big), \\ x(a) + \displaystyle\sum_{j=1}^{m} \gamma_j x(t_j) = 0, \quad t_j \in \{a, a+1, \cdots, b\}. \end{cases}$$

For a discrete domain with nonconstant step size, take $q > 1$ and consider the quantum time scale $\mathbb{T} = \{0, \cdots, q^{-2}, q^{-1}, 1, q, q^2, \cdots\}$. Then for $t \in \{qa, q^2 a, \cdots, q^{-2} b, q^{-1} b\}$, the nonlocal initial value problem (1.1) takes the form

$$\begin{cases} x(qt) = x(t) + (q-1) t f\big(t, x(qt)\big), \\ x(a) + \displaystyle\sum_{j=1}^{m} \gamma_j x(t_j) = 0, \quad t_j \in \{a, qa, \cdots, b\}. \end{cases}$$

## 2 Existence Results

In what follows, set $\|x\|_{a,b} = \max_{t \in [a,b]_{\mathbb{T}}} |x(t)|$.

**Theorem 2.1** *Assume* (1.2)–(1.5). *Then the nonlocal initial value problem* (1.1) *has at least one solution.*

*Proof:* The result will follow from the Leray-Schauder fixed point theorem once we have proven the boundedness of the set of all solutions to equations of the form $x = \lambda L x$ for $\lambda \in [0, 1]$. Let $x$ be one such solution. Then for $t \in [a, t_m]_{\mathbb{T}}$, we have

$$\begin{aligned} |x(t)| &= \lambda \left| -\alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^\sigma(s)\big)\Delta s \right. \\ &\qquad \left. + \int_a^t f\big(s, x^\sigma(s)\big)\Delta s \right| \\ &\leq \left(1 + |\alpha| \sum_{j=1}^{m} |\gamma_j|\right) \int_a^{t_m} \left| f\big(s, x^\sigma(s)\big)\right| \Delta s \\ &\leq A \int_a^{t_m} w\big(s, \|x\|_{a,t_m}\big)\Delta s. \end{aligned}$$

Now we take the maximum over $[a, t_m]_{\mathbb{T}}$ to obtain

$$\|x\|_{a,t_m} \leq A \int_a^{t_m} w\big(s, \|x\|_{a,t_m}\big)\Delta s.$$

This, according to (1.4), guarantees that

$$\|x\|_{a,t_m} \leq R_0. \tag{2.1}$$

Next, consider $t \in [t_m, b]_{\mathbb{T}}$. Then

$$\begin{aligned} |x(t)| &= \lambda \left| -\alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^\sigma(s)\big)\Delta s \right. \\ &\qquad \left. + \int_a^t f\big(s, x^\sigma(s)\big)\Delta s \right| \\ &\leq A \int_a^{t_m} w\big(s, R_0\big)\Delta s \\ &\qquad + \int_{t_m}^t p(s) q\left(|x(s)|\right)\Delta s := \varphi(t), \end{aligned}$$

and for $t \in [t_m, b]_{\mathbb{T}}$ we have $\varphi^\Delta(t) = p(t) q(|x(t)|) \leq p(t) q(\varphi(t))$. Since $\varphi$ is continuous and strictly increasing on $[t_m, b]_{\mathbb{T}}$, we have by change of variable (Theorem 5.40 in Bohner and Peterson [4]) that

$$\int_{\varphi(t_m)}^{\varphi(t)} \frac{d\eta}{q(\eta)} = \int_{t_m}^t \frac{\varphi^\Delta(s)}{q(\varphi(s))}\Delta s \leq \int_{t_m}^t p(s)\Delta s.$$

This, together with (1.5), guarantees that $\varphi(t) \leq R_1$ for all $t \in [t_m, b]_{\mathbb{T}}$ for some $R_1 > 0$. Consequently, $|x(t)| \leq R_1$ for all $t \in [t_m, b]_{\mathbb{T}}$, so that

$$\|x\|_{t_m,b} \leq R_1. \tag{2.2}$$

If we set $R := \max\{R_0, R_1\}$, then estimates (2.1) and (2.2) yield $\|x\|_{a,b} \leq R$. QED

**Remark 2.2** *If $t_m = a$, that is if we have the local initial condition $x(a) = 0$ in* (1.1), *then* (1.5) *is simply the condition* $\int_a^b p(s)\Delta s < \int_0^\infty \frac{d\eta}{q(\eta)}$.

In the next theorem we modify assumption (1.3) to get a related existence result.

**Theorem 2.3** *Assume that $f : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies for all $t \in [a,b]_{\mathbb{T}}$ and all $x \in \mathbb{R}$ the inequality*

$$|f(t,x)| \le p(t)q(|x|),$$

*with $p_0 := \int_a^b p(s)\Delta s < +\infty$ and $q : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing such that*

$$\limsup_{\eta\to\infty} \frac{\eta}{q(\eta)} > Ap_0. \tag{2.3}$$

*Then the nonlocal initial value problem* (1.1) *has at least one solution.*

*Proof:* First we show that the set $S := \{x \in \mathrm{C}[a,b]_{\mathbb{T}} : x = \lambda Lx$ for some $\lambda \in (0,1)\}$ is bounded. For any $u \in S$, we have

$$\begin{aligned}
|u(t)| &= \lambda \left| -\alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, u^\sigma(s)\big)\Delta s \right. \\
&\quad \left. + \int_a^t f\big(s, u^\sigma(s)\big)\Delta s \right| \\
&\le |\alpha| \sum_{j=1}^{m} |\gamma_j| \int_a^{t_j} p(s) q\left(|u^\sigma(s)|\right)\Delta s \\
&\quad + \int_a^b p(s) q\left(|u^\sigma(s)|\right)\Delta s \\
&\le \left( |\alpha| \sum_{j=1}^{m} |\gamma_j| \int_a^{t_j} p(s)\Delta s \right. \\
&\quad \left. + \int_a^b p(s)\Delta s \right) q\left(\|u\|_{a,b}\right).
\end{aligned}$$

Let $\eta_0 = \|u\|_{a,b}$. Then

$$\begin{aligned}
\eta_0 &\le \left( |\alpha| \sum_{j=1}^{m} |\gamma_j| \int_a^{t_j} p(s)\Delta s \right. \\
&\quad \left. + \int_a^b p(s)\Delta s \right) q(\eta_0) \\
&\le \left(1 + |\alpha| \sum_{j=1}^{m} |\gamma_j|\right) \left(\int_a^b p(s)\Delta s\right) q(\eta_0) \\
&= Ap_0 q(\eta_0).
\end{aligned}$$

Hence

$$\frac{\eta_0}{q(\eta_0)} \le Ap_0. \tag{2.4}$$

On the other hand, the conditions on $q$ and $p_0$ imply that there exists $\eta^* > 0$ such that for all $\eta > \eta^*$ we have

$$\frac{\eta}{q(\eta)} > Ap_0. \tag{2.5}$$

Comparing the last two inequalities we see that $\eta_0 \le \eta^*$. Consequently,

$$\|u\|_{a,b} \le \eta^*.$$

This shows that the set $S$ is bounded. By the Schaefer fixed point theorem (see Smart [10]) the equation $x = \lambda Lx$ has a solution for $\lambda = 1$, which is a solution of (1.1). QED

# 3 Some Particular Cases

## 3.1 Nonlinearities with at most linear growth

In this subsection we show that the existence of solutions to problem (1.1) follows directly from the Schauder fixed point theorem in the particular case that (1.2) again holds, and the nonlinearity $f$ satisfies the following growth condition in $x$,

$$|f(t,x)| \le \begin{cases} k|x| + d: & t \in [a,t_m]_{\mathbb{T}} \\ c|x| + d: & t \in [t_m,b]_{\mathbb{T}} \end{cases} \tag{3.1}$$

for all $x \in \mathbb{R}$, provided that $k(t_m - a)A < 1$. In this case (1.3) holds, with $w(t,\eta) = k\eta + d$, $p(t) = 1$, $q(\eta) = c\eta + d$, and

$$R_0 = \frac{k(t_m - a)A}{\big(1 - k(t_m - a)A\big)}.$$

In order to apply the Schauder fixed point theorem, we seek a nonempty, bounded, closed and convex subset $B$ of $\mathrm{C}[a,b]_{\mathbb{T}}$ with $L(B) \subset B$. Let $x$ be any element of $\mathrm{C}[a,b]_{\mathbb{T}}$. For $t \in [a,t_m]_{\mathbb{T}}$, we have

$$\begin{aligned}
|Lx(t)| &= \left| -\alpha \sum_{j=1}^{m} \gamma_j \int_a^{t_j} f\big(s, x^\sigma(s)\big)\Delta s \right. \\
&\quad \left. + \int_a^t f\big(s, x^\sigma(s)\big)\Delta s \right| \\
&\le A \int_a^{t_m} \big|f\big(s, x^\sigma(s)\big)\big|\Delta s \\
&\le k(t_m - a)A\|x\|_{a,t_m} + d(t_m - a)A.
\end{aligned}$$

As a result,

$$\|Lx\|_{a,t_m} \le k(t_m - a)A\|x\|_{a,t_m} + d(t_m - a)A. \tag{3.2}$$

For any constant $\beta > 0$, recall the delta exponential function (see Definition 2.30 in Bohner and Peterson [3]) given by $e_\beta(t,t_m)$; then for any $t\in[t_m,b]_{\mathbb{T}}$ we have

$$\begin{aligned}|Lx(t)| &= \left|-\alpha\sum_{j=1}^{m}\gamma_j\int_a^{t_j} f\big(s,x^\sigma(s)\big)\Delta s \right.\\ &\quad \left. +\int_a^t f\big(s,x^\sigma(s)\big)\Delta s\right|\\ &\le k(t_m-a)A\|x\|_{a,t_m}+d(t_m-a)A\\ &\quad +\int_{t_m}^t (c|x(s)|+d)\,\Delta s\\ &\le k(t_m-a)A\|x\|_{a,t_m}\\ &\quad +d\big((t_m-a)A+b-t_m\big)\\ &\quad +c\int_{t_m}^t e_{\ominus\beta}(s,t_m)|x(s)|e_\beta(s,t_m)\Delta s\\ &\le k(t_m-a)A\|x\|_{a,t_m}+d_0+\frac{c}{\beta}e_\beta(t,t_m)\|x\|_\beta,\end{aligned}$$

where $d_0:=d\big((t_m-a)A+b-t_m\big)$, and the Bielecki-type norm introduced by Tisdell and Zaidi [14] on time scales denoted $\|x\|_\beta$ is given by

$$\|x\|_\beta:=\sup_{t\in[t_m,b]_{\mathbb{T}}} e_{\ominus\beta}(t,t_m)|x(t)|.$$

Multiplying by $e_{\ominus\beta}(t,t_m)$ and taking the supremum over $[t_m,b]_{\mathbb{T}}$, we arrive at

$$\|Lx\|_\beta\le k(t_m-a)A\|x\|_{a,t_m}+d_0+\frac{c}{\beta}\|x\|_\beta. \tag{3.3}$$

If we consider an equivalent norm on $\mathrm{C}[a,b]_{\mathbb{T}}$ given by

$$\|x\|:=\max\left\{\|x\|_{a,t_m},\|x\|_\beta\right\},$$

then from (3.2) and (3.3) we have

$$\|Lx\|\le\big(k(t_m-a)A+c/\beta\big)\|x\|+d_1, \tag{3.4}$$

where $d_1:=\max\{d_0,d(t_m-a)A\}$. Since $k(t_m-a)A<1$, there exists a $\beta>0$ large enough such that $k(t_m-a)A+c/\beta<1$. Hence there exists a number $R>0$ with

$$\big(k(t_m-a)A+c/\beta\big)R+d_1\le R. \tag{3.5}$$

Now we take $B=\{x\in\mathrm{C}[a,b]_{\mathbb{T}}:\|x\|\le R\}$. Inequalities (3.4) and (3.5) guarantee that $L(B)\subset B$, and thus the Schauder fixed point theorem can be applied.

### 3.2 Lipschitz nonlinearities

We deal in this subsection with problem (1.1) when the nonlinearity $f$ satisfies a Lipschitz condition in $x$ of the form

$$|f(t,x)-f(t,y)|\le\ell(t)|x-y|,\quad t\in[a,b]_{\mathbb{T}},\quad x,y\in\mathbb{R}, \tag{3.6}$$

where $\ell:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is a bounded function. The following existence and uniqueness result is based on the Banach contraction principle.

**Theorem 3.1** *Assume* (1.2) *and that $f(\cdot,x)$ is measurable for all $x\in\mathbb{R}$. Assume moreover that $f(\cdot,0)$ is bounded, and that there exists a bounded function $\ell$ such that* (3.6) *holds, with*

$$A(t_m-a)\|\ell\|_{a,t_m}<1.$$

*Then* (1.1) *has a unique solution $x^*\in\mathrm{C}[a,b]_{\mathbb{T}}$ such that $\|x_n-x^*\|_{a,b}\to0$ as $n\to\infty$, where $x_0$ is any function in $\mathrm{C}[a,b]_{\mathbb{T}}$ and $x_n=Lx_{n-1}$ for each $n\in\mathbb{N}$.*

*Proof:* Let $x,y\in\mathrm{C}[a,b]_{\mathbb{T}}$. For $t\in[a,t_m]_{\mathbb{T}}$ we have

$$\begin{aligned}|Lx(t)-Ly(t)| &\le \|\ell\|_{a,t_m}A\int_a^{t_m}|x(s)-y(s)|\Delta s\\ &\le \|\ell\|_{a,t_m}A(t_m-a)\|x-y\|_{a,t_m}.\end{aligned}$$

It follows that

$$\|Lx-Ly\|_{a,t_m}\le\|\ell\|_{a,t_m}A(t_m-a)\|x-y\|_{a,t_m}. \tag{3.7}$$

For $t\in[t_m,b]_{\mathbb{T}}$ we have

$$\begin{aligned}|Lx(t)-Ly(t)| &\le \|\ell\|_{a,t_m}A(t_m-a)\|x-y\|_{a,t_m}\\ &\quad +\|\ell\|_{t_m,b}\int_{t_m}^t|x(s)-y(s)|\Delta s\\ &\le \|\ell\|_{a,t_m}A(t_m-a)\|x-y\|_{a,t_m}\\ &\quad +\frac{\|\ell\|_{t_m,b}}{\beta}e_\beta(t,t_m)\|x-y\|_\beta.\end{aligned}$$

Consequently,

$$\begin{aligned}\|Lx-Ly\|_\beta &\le \|\ell\|_{a,t_m}A(t_m-a)\|x-y\|_{a,t_m}\\ &\quad +\frac{\|\ell\|_{t_m,b}}{\beta}\|x-y\|_\beta,\end{aligned}$$

so that

$$\|Lx-Ly\|\le\left(\|\ell\|_{a,t_m}A(t_m-a)+\frac{\|\ell\|_{t_m,b}}{\beta}\right)\|x-y\|.$$

Finally, we choose any $\beta>0$ such that

$$\|\ell\|_{a,t_m}A(t_m-a)+\|\ell\|_{t_m,b}/\beta<1,$$

and we apply the Banach contraction principle. QED

**Remark 3.2** *Under the assumptions of Theorem* 3.1*, condition* (3.1) *is satisfied with $k=\|\ell\|_{a,t_m}$, $c=\|\ell\|_{t_m,b}$, and $d=\|f(\cdot,0)\|_{a,b}$.*

## 4 Example

In this section we present an example applying Theorem 2.1.

**Example 4.1** *For $\mathbb{T}=\mathbb{R}$ and $[a,b]=[0,1]$, the nonlocal initial value problem*

$$\begin{cases}x'(t)=f\big(t,x(t)\big), & t\in(0,1),\\ x(0)+\frac{1}{2}x\left(1/3\right)+\frac{1}{4}x\left(2/3\right)=0,\end{cases} \tag{4.1}$$

*where*

$$f(t,x) = \sin\big(6\pi t\big) \begin{cases} 1 & : t \in [0, 2/3], \\ x^2 & : t \in [2/3, 1], \end{cases}$$

*has at least one solution.*

*Proof:* Considering the function $f$ and its bounds,

$$|f(t,x)| \le \begin{cases} w\big(t,|x|\big) = |\sin(6\pi t)| \cdot 1 & : t \in [0,2/3], \\ p(t)q\big(|x|\big) = |\sin(6\pi t)| \cdot x^2 & : t \in [2/3, 1]. \end{cases}$$

It is straightforward to calculate the constant values

$$\alpha = (1 + 1/2 + 1/4)^{-1} = 4/7,$$

$A = \frac{10}{7}$, and $R_0^* = \frac{25}{14\pi}$. Clearly if we take $R_0 = \frac{40}{21\pi} > 0$, it follows that $\eta > R_0$ implies

$$\begin{aligned} \frac{1}{\eta}\int_0^{2/3} w(s,\eta)ds &= \frac{1}{\eta}\int_0^{2/3} |\sin(6\pi s)|ds = \frac{4}{3\pi\eta} \\ &< \frac{1}{A} = \frac{7}{10}, \end{aligned}$$

and

$$\begin{aligned} \int_{2/3}^1 p(s)ds &= \int_{2/3}^1 |\sin(6\pi s)|ds = \frac{2}{3\pi} \\ &< \int_{R_0^*}^\infty \frac{d\eta}{q(\eta)} = \int_{R_0^*}^\infty \frac{d\eta}{\eta^2} = \frac{1}{R_0^*} = \frac{14\pi}{25}. \end{aligned}$$

Thus (1.2)−(1.5) are all satisfied, so that by Theorem 2.1, the nonlocal initial value problem (4.1) has at least one solution. QED

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