Applied Mathematics Letters 22 (2009) 1591–1597

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Nonlinear oscillation of second-order dynamic equations on time scales

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ARTICLE INFO

Article history:
Received 1 May 2008
Received in revised form 8 May 2009
Accepted 12 May 2009

Keywords:
Time scales
Oscillation
Riccati substitution
Second order

ABSTRACT

Interval oscillation criteria are established for a second-order nonlinear dynamic equation on time scales by utilizing a generalized Riccati technique and the Young inequality. The theory can be applied to second-order dynamic equations regardless of the choice of delta or nabla derivatives.

1. Introduction

In the wake of S. Hilger's breakthrough introduction of measure chains [1], a steadily diversifying body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time-scale calculus, where a time scale (measure chain) is simply any nonempty closed set of real numbers. We illustrate this expansive understanding by extending some continuous results from differential equations to dynamic equations on arbitrary time scales (unbounded above), thus including as corollaries difference equations and q-difference equations. In particular, we are concerned with the oscillatory behavior of the second-order nonlinear dynamic equation given by

$$\left(r(t)\Phi_\alpha(x^\Delta(t))\right)^\Delta + f\left(t, x^\sigma(t), x^\Delta(t)\right) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0 \in \mathbb{T}$, $\Phi_\alpha(x) = |x|^{\alpha-1}x$ with $\alpha > 0$, the function $r : \mathbb{T} \to (0, \infty)_{\mathbb{R}}$ is right-dense continuous, and the function $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is right-dense continuous in the time-scale variable. We will suppose that the functions r and f are sufficiently smooth to ensure that (1.1) always has solutions that are continuable on all of $[t_0, \infty)_{\mathbb{T}}$. Throughout this work a knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [2,3] and Kaymakçalan, Lakshmikantham, and Sivasundaram [4].

Eq. (1.1) is studied extensively by Wang [5] for the case when $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$; see also Güvenilir and Zafer [6]. Related discussions can be found by Nasr [7] and Wong [8], who study the oscillation of

$$(rx')'(t) + p(t)|x(t)|^{\alpha-1}x(t) = f(t),$$

and an extension of this work by Sun [9] to the equation

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = f(t).$$

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doi:10.1016/j.aml.2009.05.004

In a related work, Çakmak and Tiryaki [10] consider oscillation for the forced equation

$$x''(t) + p(t)f(x(\tau(t))) = a(t).$$

See also Li and Zhu [11], and Wong [12].

Recently there have also been numerous papers on second-order nonlinear dynamic equations on time scales. For a sampling of the work done on second-order equations, see the monograph by Agarwal, Grace, and O'Regan [13]. For a few examples of work since then, Bohner and Tisdell [14] examine oscillation and nonoscillation for

$$\left(rx^{\Delta}\right)^{\Delta}(t) + p(t)x^{\sigma}(t) = f(t).$$

Erbe, Peterson, and Saker [15] study the unforced delay dynamic equation

$$\left(rx^{\Delta}\right)^{\Delta}(t) + p(t)x(\tau(t)) = 0,$$

and Saker [16] studies the oscillation of the related forced dynamic equation

$$\left(rx^{\Delta}\right)^{\Delta}(t) + p(t)f(x^{\sigma}(t)) = a(t).$$

In [17,18] the authors consider the oscillation of the neutral delay dynamic equation

$$\left(r(t)([x(t) + p(t)x(\tau(t))]^{\Delta})^{\gamma}\right)^{\Delta} + f(t, x(\delta(t))) = 0,$$

while Agarwal, O'Regan, and Saker [19] discuss the oscillatory behavior of the nonlinear perturbed dynamic equation

$$\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} + f(t, x(t)) = g(t, x(t), x^{\Delta}(t)).$$

Oscillatory criteria for the forced dynamic equation

$$\left(rx^{\Delta}\right)^{\Delta}(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) + q(t)|x(\theta(t))|^{\beta-1}x(\theta(t)) = f(t)$$

are analyzed in [20]. Extension of these results to equations of the form

$$\left(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)\right)^{\Delta} + p(t)|x(\tau(t))|^{\beta-1}x(\tau(t)) + q(t)|x(\theta(t))|^{\gamma-1}x(\theta(t)) = f(t)$$

is given in [21].

Motivated by these previous results, we herein utilize a generalized Riccati technique and the well-known Young inequality to ascertain several new interval criteria for the oscillation of (1.1), in other words criteria given by the behavior of (1.1) on only a sequence of subintervals of $[t_0, \infty)_{\mathbb{T}}$. These criteria unify, extend, and generalize to arbitrary time scales a number of extant results.

2. Foundational lemmas

A solution of (1.1) is a nontrivial real function $x : \mathbb{T} \to \mathbb{R}$ such that $x \in C^{\Delta}_{rd}[T, \infty)_{\mathbb{T}}$ and $r\Phi_{\alpha}(x^{\Delta}) \in C^{\Delta}_{rd}[T, \infty)_{\mathbb{T}}$ for all $T \in \mathbb{T}$ with $T \geq t_0$, and x satisfies (1.1). A function x is an oscillatory solution of (1.1) if and only if x is a solution of (1.1) that is neither eventually positive nor eventually negative. Eq. (1.1) is oscillatory if and only if every solution x of (1.1) is oscillatory.

Following [5,6,20], define for $a, b \in [t_0, \infty)_{\mathbb{T}}$ with $a < b$ the admissible set

$$J(a, b) := \left\{u \in C^{\Delta}_{rd}[a, b]_{\mathbb{T}} : u(a) = 0 = u(b), u \not\equiv 0\right\}.$$

Lemma 2.1. *If x is a solution of (1.1) such that the product $xx^{\sigma} > 0$ on some interval $[a, b]_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$, then we have the strict inequality*

$$\int_a^b \left[|u^{\sigma}(t)|^{\alpha+1} \frac{f\left(t, x^{\sigma}(t), x^{\Delta}(t)\right)}{\Phi_{\alpha}(x^{\sigma}(t))} - r(t)\left|u^{\Delta}(t)\right|^{\alpha+1} \right] \Delta t < 0 \tag{2.1}$$

for any $u \in J(a, b)$.

Proof. Define the Riccati substitution w via

$$w(t) := -r(t)\frac{\Phi_{\alpha}(x^{\Delta}(t))}{\Phi_{\alpha}(x(t))}, \quad t \in [a, b]_{\mathbb{T}}.$$

Using the delta quotient rule and the fact that x is a solution of (1.1), we have

$$w^{\Delta}(t) = \frac{r(t)\Phi(x^{\Delta}(t))\,[\Phi_{\alpha}(x(t))]^{\Delta}}{\Phi_{\alpha}(x(t))\Phi_{\alpha}(x^{\sigma}(t))} + \frac{f\left(t, x^{\sigma}(t), x^{\Delta}(t)\right)}{\Phi_{\alpha}(x^{\sigma}(t))}. \tag{2.2}$$

Let $u \in J(a, b)$; as

$$(u\Phi_\alpha(u)w)^\Delta = u^\sigma \Phi_\alpha(u^\sigma)w^\Delta + (|u|^{\alpha+1})^\Delta w, \tag{2.3}$$

we have

$$(u\Phi_\alpha(u)w)^\Delta = |u^\sigma|^{\alpha+1}\frac{f\left(t, x^\sigma(t), x^\Delta(t)\right)}{\Phi_\alpha(x^\sigma(t))} - |u^\Delta|^{\alpha+1}r + G(u, w), \tag{2.4}$$

where

$$G(u, w) = |u^\Delta|^{\alpha+1}r - (|u|^{\alpha+1})^\Delta w + |u^\sigma|^{\alpha+1}\frac{r\,\Phi_\alpha(x^\Delta)(\Phi_\alpha(x))^\Delta}{\Phi_\alpha(x)\Phi_\alpha(x^\sigma)}. \tag{2.5}$$

On the other hand it can be shown that $G(u, w) \geq 0$, and that $G(u, w) = 0$ if and only if

$$u^\Delta = \Phi_\alpha^{-1}(w/r)u. \tag{2.6}$$

The proof consists of two parts depending on whether t is a right-dense or a right-scattered point. Young's inequality is employed when t is right-dense, while differential calculus is used when t is right-scattered. For a complete discussion we refer the reader to [22, Theorem 3.3]; see also [21] for a sketch of the proof.

Now since $1 + \mu\Phi_\alpha^{-1}(w/r) > 0$, Eq. (2.6) has a unique solution satisfying $u(a) = 0$. Clearly, the unique solution is $u \equiv 0$. Therefore, $G(u, w) > 0$ on $[a, b]_\mathbb{T}$. Integrating the inequality (2.4) from a to b and using the fact that $u(a) = 0 = u(b)$, we obtain

$$\int_a^b \left\{|u^\sigma(t)|^{\alpha+1}\frac{f\left(t, x^\sigma(t), x^\Delta(t)\right)}{\Phi_\alpha(x^\sigma(t))} - |u^\Delta(t)|^{\alpha+1}r(t)\right\}\Delta t < 0.$$

Thus, (2.1) holds. □

Related to (1.1) is the dynamic equation with mixed delta and nabla derivatives

$$\left(r\Phi_\alpha(x^\Delta)\right)^\nabla(t) + f\left(t, x(t), x^\Delta(t)\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T}. \tag{2.7}$$

A solution of (2.7) is a nontrivial real function $x : \mathbb{T} \to \mathbb{R}$ such that $x \in C^\Delta[T, \infty)_\mathbb{T}$ and $r\Phi_\alpha(x^\Delta) \in C_{ld}^\nabla[T, \infty)_\mathbb{T}$ for all $T \in \mathbb{T}$ with $T \geq t_0$, and x satisfies (2.7). Note that in this case a solution x must satisfy the stronger condition of $x \in C^\Delta[T, \infty)_\mathbb{T}$ rather than the previous case of $x \in C_{rd}^\Delta[T, \infty)_\mathbb{T}$; to use the formula $x^{\Delta\rho} = x^\nabla$ we need x^Δ to be continuous.

We will show that time-scale modifications of the previous arguments from Lemma 2.1 lead to completely parallel results for (2.7). To this end, we begin by denoting for $a, b \in [t_0, \infty)_\mathbb{T}$ with $a < b$ the admissible set

$$L(a, b) := \left\{u \in C_{ld}^\nabla[a, b]_\mathbb{T} : u(a) = 0 = u(b),\ u \not\equiv 0\right\}.$$

Then we have the following lemma corresponding to Lemma 2.1, where the technique is similar to that found in [20].

Lemma 2.2. *If x is a solution of (2.7) such that $x^\rho x > 0$ on some interval $[a, b]_\mathbb{T} \subset [t_0, \infty)_\mathbb{T}$, then we have the strict inequality*

$$\int_a^b \left[|u(t)|^{\alpha+1}\frac{f\left(t, x(t), x^\Delta(t)\right)}{\Phi_\alpha(x(t))} - r^\rho(t)\left|u^\nabla(t)\right|^{\alpha+1}\right]\nabla t < 0 \tag{2.8}$$

for any $u \in L(a, b)$.

Proof. The proof is similar to that of Lemma 2.1. □

Analogous lemmas can be stated for related equations of the form

$$\left(r\Phi_\alpha(x^\nabla)\right)^\nabla(t) + f\left(t, x^\rho(t), x^\nabla(t)\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T}$$

and

$$\left(r\Phi_\alpha(x^\nabla)\right)^\Delta(t) + f\left(t, x(t), x^\nabla(t)\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T}.$$

3. The main results

We return to our consideration of (1.1). The structure and content of this section are largely motivated by the special case when $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, found in Wang [5].

Theorem 3.1. *If for some interval $[a, b]_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$ there exists an admissible function $u \in J(a, b)$ such that for any $x \in C^{\Delta}_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $xx^{\sigma} > 0$ on $[a, b]_{\mathbb{T}}$ we have*

$$\int_a^b \left[|u^{\sigma}(t)|^{\alpha+1} \frac{f\left(t, x^{\sigma}(t), x^{\Delta}(t)\right)}{\Phi_{\alpha}(x^{\sigma}(t))} - r(t)|u^{\Delta}(t)|^{\alpha+1} \right] \Delta t \geq 0, \tag{3.1}$$

then every solution of (1.1) has at least one generalized zero in $[a, b]_{\mathbb{T}}$.

Proof. Suppose there exists a solution x of (1.1) such that $xx^{\sigma} > 0$ on $[a, b]_{\mathbb{T}}$. Then by assumption there exists a $u \in J(a, b)$ such that (3.1) holds for this particular solution x. From Lemma 2.1, though, we then have that (2.1) holds, a contradiction of (3.1). □

Theorem 3.2. *If for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist $a < b \in [T, \infty)_{\mathbb{T}}$ and $u \in J(a, b)$ such that for any $x \in C^{\Delta}_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $xx^{\sigma} > 0$ on $[a, b]_{\mathbb{T}}$ the inequality (3.1) holds, then (1.1) is oscillatory.*

Proof. Select an increasing sequence of time-scale points $\{T_j\}_{j=1}^{\infty} \subset [t_0, \infty)_{\mathbb{T}}$ such that $T_j \to \infty$ as $j \to \infty$. By supposition, for each $j \in \mathbb{N}$ there exist $a_j, b_j \in [t_0, \infty)_{\mathbb{T}}$ with $T_j \leq a_j < b_j$ such that (3.1) holds, where the a and b are replaced by a_j and b_j, respectively. From Theorem 3.1 every solution x of (1.1) has at least one generalized zero t_j in $[a_j, b_j]_{\mathbb{T}}$. Since $T_j \leq a_j \leq t_j$ for each $j \in \mathbb{N}$, every solution has arbitrarily large generalized zeros. Hence, (1.1) is oscillatory. □

With selective choices of the function $f(t, y, z)$, using Theorem 3.2 one can derive a number of oscillation criteria that unify, extend, and generalize a number of existing results. The following corollary is one such case.

Corollary 3.3. *Suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist time-scale points $a_1 < b_1 \in [T, \infty)_{\mathbb{T}}$ and $a_2 < b_2 \in [b_1, \infty)_{\mathbb{T}}$, functions $p, q \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $p(t) \geq 0$ for $t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}$, $(-1)^k q(t) \geq 0$ for $t \in [a_k, b_k]_{\mathbb{T}}$, $k = 1, 2$, and*

$$yf(t, y, z) \geq p(t)|y|^{\alpha+1} - q(t)y, \quad t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \; y \neq 0, \; z \in \mathbb{R}.$$

If there exists $u \in J(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} \left[|u^{\sigma}(t)|^{\alpha+1} p(t) - r(t) \left|u^{\Delta}(t)\right|^{\alpha+1} \right] \Delta t \geq 0, \quad i = 1, 2,$$

then (1.1) is oscillatory.

Corollary 3.4. *Suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist time-scale points $a_1 < b_1 \in [T, \infty)_{\mathbb{T}}$ and $a_2 < b_2 \in [b_1, \infty)_{\mathbb{T}}$, functions $p, q \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $p(t) \geq 0$ for $t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}$, $\beta > \alpha$, q has different signs on $[a_1, b_1]_{\mathbb{T}}$ and $[a_2, b_2]_{\mathbb{T}}$, and*

$$yf(t, y, z) \geq p(t)|y|^{\beta+1} - q(t)y, \quad t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \; y \neq 0, \; z \in \mathbb{R}.$$

If there exists $u \in J(a_i, b_i)$ such that

$$\beta(\beta - \alpha)^{\alpha/\beta - 1} \alpha^{-\alpha/\beta} \int_{a_i}^{b_i} |u^{\sigma}(t)|^{\alpha+1} p(t)^{\alpha/\beta} |q(t)|^{1-\alpha/\beta} \Delta t \geq \int_{a_i}^{b_i} r(t) \left|u^{\Delta}(t)\right|^{\alpha+1} \Delta t, \quad i = 1, 2,$$

then (1.1) is oscillatory.

Proof. On using the inequality

$$y^{\lambda} z^{1-\lambda} \leq \lambda y + (1 - \lambda) z, \quad y, z \in [0, \infty)_{\mathbb{R}}, \; \lambda \in (0, 1)_{\mathbb{R}},$$

we obtain

$$p(t)|y|^{\beta+1} - q(t)y \geq \beta(\beta - \alpha)^{\alpha/\beta - 1} \alpha^{-\alpha/\beta} p(t)^{\alpha/\beta} |q(t)|^{1-\alpha/\beta} |y|^{\alpha+1},$$

where $q(t)y \geq 0$ is arranged by working on the right interval(s). The proof is complete in view of Corollary 3.3. □

Theorem 3.1 is proved with the help of Lemma 2.1. Making use of Lemma 2.2 we easily obtain the following analogous results.

Theorem 3.5. *If for some interval $[a, b]_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$ there exists an admissible function $u \in L(a, b)$ such that for any $x \in C^{\Delta}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $x^{\rho} x > 0$ on $[a, b]_{\mathbb{T}}$ we have*

$$\int_a^b \left[|u(t)|^{\alpha+1} \frac{f\left(t, x(t), x^{\Delta}(t)\right)}{\Phi_{\alpha}(x(t))} - r^{\rho}(t) \left|u^{\nabla}(t)\right|^{\alpha+1} \right] \nabla t \geq 0, \tag{3.2}$$

then every solution of (2.7) has at least one generalized zero in $[a, b]_{\mathbb{T}}$.

Theorem 3.6. *If for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist $a < b \in [T, \infty)_{\mathbb{T}}$ and $u \in L(a, b)$ such that for any $x \in C^{\Delta}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ with $x^{\rho}x > 0$ on $[a, b]_{\mathbb{T}}$ the inequality (3.2) holds, then (2.7) is oscillatory.*

Corollary 3.7. *Suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist time-scale points $a_1 < b_1 \in [T, \infty)_{\mathbb{T}}$ and $a_2 < b_2 \in [b_1, \infty)_{\mathbb{T}}$, functions $p, q \in C_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $p(t) \ge 0$ for $t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}$, $(-1)^k q(t) \ge 0$ for $t \in [a_k, b_k]_{\mathbb{T}}$, $k = 1, 2$, and*

$$yf(t, y, z) \ge p(t)|y|^{\alpha+1} - q(t)y, \quad t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}},\ y \ne 0,\ z \in \mathbb{R}.$$

If there exists $u \in L(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} \left[|u(t)|^{\alpha+1} p(t) - r^{\rho}(t) \left|u^{\nabla}(t)\right|^{\alpha+1} \right] \nabla t \ge 0, \quad i = 1, 2,$$

then (2.7) is oscillatory.

Corollary 3.8. *Suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist time-scale points $a_1 < b_1 \in [T, \infty)_{\mathbb{T}}$ and $a_2 < b_2 \in [b_1, \infty)_{\mathbb{T}}$, functions $p, q \in C_{ld}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $p(t) \ge 0$ for $t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}$, $\beta > \alpha$, q has different signs on $[a_1, b_1]_{\mathbb{T}}$ and $[a_2, b_2]_{\mathbb{T}}$, and*

$$yf(t, y, z) \ge p(t)|y|^{\beta+1} - q(t)y, \quad t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}},\ y \ne 0,\ z \in \mathbb{R}.$$

If there exists $u \in L(a_i, b_i)$ such that

$$\beta(\beta - \alpha)^{\alpha/\beta - 1} \alpha^{-\alpha/\beta} \int_{a_i}^{b_i} |u(t)|^{\alpha+1} p(t)^{\alpha/\beta} |q(t)|^{1-\alpha/\beta} \nabla t \ge \int_{a_i}^{b_i} r^{\rho}(t) \left|u^{\nabla}(t)\right|^{\alpha+1} \nabla t, \quad i = 1, 2,$$

then (2.7) is oscillatory.

It should be remarked that one may take $q \equiv 0$ in Corollaries 3.3 and 3.7, but $q \equiv 0$ is not allowed in Corollaries 3.4 and 3.8. In fact, the problem when $q \equiv 0$ is open for any time scale. Another open problem that is of both theoretical and practical interest is to establish interval oscillation criteria for the case $0 < \beta < \alpha$.

4. Delay dynamic equations

Consider the delay dynamic equation

$$\left(r(t)\Phi_{\alpha}(x^{\Delta}(t))\right)^{\Delta} + f\left(t, x(g(t)), x^{\Delta}(t)\right) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.1}$$

where $f(t, y, z) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is right-dense continuous in the time-scale variable t and nondecreasing in y, $g : \mathbb{T} \to \mathbb{T}$ is nondecreasing right-dense continuous function such that $g(t) \le t$ with $\lim_{t\to\infty} g(t) = \infty$.

Lemma 4.1. *Let x be a solution of (4.1) such that the product $xx^{\sigma} > 0$ on some interval $[a, b]_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$. If*

$$yf(t, y, z) > 0, \quad t \in [a, b]_{\mathbb{T}},\ y \ne 0,\ z \in \mathbb{R}, \tag{4.2}$$

then we have the strict inequality

$$\int_a^b \left[|u^{\sigma}(t)|^{\alpha+1} \frac{f\left(t, x^{\sigma}(t)G(t), x^{\Delta}(t)\right)}{\Phi_{\alpha}(x^{\sigma}(t))} - r(t)\left|u^{\Delta}(t)\right|^{\alpha+1} \right] \Delta t < 0 \tag{4.3}$$

for any $u \in J(a, b)$, where

$$G(t) = \frac{g(t) - g(a)}{\sigma(t) - g(a)}.$$

Proof. Proceeding as in the proof of Lemma 2.1, we obtain, in a similar manner to (2.4), the inequality

$$(u\Phi_{\alpha}(u)w)^{\Delta} \ge |u^{\sigma}|^{\alpha+1} \frac{f\left(t, x(g(t)), x^{\Delta}(t)\right)}{\Phi_{\alpha}(x^{\sigma}(t))} - |u^{\Delta}|^{\alpha+1} r + G(u, w), \tag{4.4}$$

where $G(u, w)$ is given in (2.5). As in [20,21], one can easily see that

$$\frac{x(g(t))}{x^{\sigma}(t)} \ge G(t), \quad t \in (a, b]_{\mathbb{T}}. \tag{4.5}$$

Using (4.5) in (4.4) we have

$$(u\Phi_\alpha(u)w)^\Delta \geq |u^\sigma|^{\alpha+1}\frac{f\left(t, x^\sigma(t)G(t), x^\Delta(t)\right)}{\Phi_\alpha(x^\sigma(t))} - |u^\Delta|^{\alpha+1}r + G(u, w). \tag{4.6}$$

The rest of the proof is the same as that given for Lemma 2.1. □

Theorem 4.2. *If for some interval $[a, b]_\mathbb{T} \subset [t_0, \infty)_\mathbb{T}$ f satisfies (4.2) and there exists an admissible function $u \in J(a, b)$ such that for any $x \in C^\Delta_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ with $xx^\sigma > 0$ on $[a, b]_\mathbb{T}$ we have*

$$\int_a^b \left[|u^\sigma(t)|^{\alpha+1}\frac{f\left(t, x^\sigma(t)G(t), x^\Delta(t)\right)}{\Phi_\alpha(x^\sigma(t))} - r(t)\left|u^\Delta(t)\right|^{\alpha+1}\right]\Delta t \geq 0, \tag{4.7}$$

then every solution of (4.1) has at least one generalized zero in $[a, b]_\mathbb{T}$.

Corollary 4.3. *Suppose that for any $T \in [t_0, \infty)_\mathbb{T}$ there exist time-scale points $a_1 < b_1 \in [T, \infty)_\mathbb{T}$ and $a_2 < b_2 \in [b_1, \infty)_\mathbb{T}$, functions $p, q \in C_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ such that $p(t) \geq 0$ for $t \in [a_1, b_1]_\mathbb{T} \cup [a_2, b_2]_\mathbb{T}$, $(-1)^k q(t) \geq 0$ for $t \in [a_k, b_k]_\mathbb{T}$, $k = 1, 2$, and*

$$yf(t, y, z) \geq p(t)|y|^{\alpha+1} - q(t)y, \quad t \in [a_1, b_1]_\mathbb{T} \cup [a_2, b_2]_\mathbb{T},\ y \neq 0,\ z \in \mathbb{R}.$$

If there exists $u \in J(a_i, b_i)$ such that

$$\int_{a_i}^{b_i}\left[|u^\sigma(t)|^{\alpha+1}p(t)G(t)^\alpha - r(t)\left|u^\Delta(t)\right|^{\alpha+1}\right]\Delta t \geq 0, \quad i = 1, 2,$$

then (4.1) is oscillatory.

Corollary 4.4. *Suppose that for any $T \in [t_0, \infty)_\mathbb{T}$ there exist time-scale points $a_1 < b_1 \in [T, \infty)_\mathbb{T}$ and $a_2 < b_2 \in [b_1, \infty)_\mathbb{T}$, functions $p, q \in C_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ such that $p(t) \geq 0$ for $t \in [a_1, b_1]_\mathbb{T} \cup [a_2, b_2]_\mathbb{T}$, $\beta > \alpha$, q has different signs on $[a_1, b_1]_\mathbb{T}$ and $[a_2, b_2]_\mathbb{T}$, and*

$$yf(t, y, z) \geq p(t)|y|^{\beta+1} - q(t)y, \quad t \in [a_1, b_1]_\mathbb{T} \cup [a_2, b_2]_\mathbb{T},\ y \neq 0,\ z \in \mathbb{R}.$$

If there exists $u \in J(a_i, b_i)$ such that

$$\beta(\beta-\alpha)^{\alpha/\beta-1}\alpha^{-\alpha/\beta}\int_{a_i}^{b_i}|u^\sigma(t)|^{\alpha+1}p(t)^{\alpha/\beta}G(t)^\alpha|q(t)|^{1-\alpha/\beta}\Delta t \geq \int_{a_i}^{b_i} r(t)\left|u^\Delta(t)\right|^{\alpha+1}\Delta t, \quad i = 1, 2,$$

then (4.1) is oscillatory.

Analogous results can be stated for related delay dynamic equations of the form

$$\left(r\Phi_\alpha(x^\Delta)\right)^\nabla(t) + f\left(t, x(g(t)), x^\Delta(t)\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T},$$
$$\left(r\Phi_\alpha(x^\nabla)\right)^\nabla(t) + f\left(t, x(g(t)), x^\nabla(t)\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T},$$

and

$$\left(r\Phi_\alpha(x^\nabla)\right)^\Delta(t) + f\left(t, x(g(t)), x^\nabla(t)\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T}.$$

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