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Global asymptotic behavior for delay dynamic equations

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Abstract

We give conditions under which the trivial solution of a first-order nonlinear variable-delay dynamic equation is asymptotically stable, for arbitrary time scales that are unbounded above. In an example, we apply our techniques to a logistic dynamic equation on isolated, unbounded time scales.

MSC: 39A10; 34B10

Keywords: Time scales; Asymptotic behavior; Nonlinear equation; Delay

1. Nonlinear variable-delay dynamic equation

In the wake of Hilger's landmark paper [1], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time-scale calculus, where a time scale is simply any nonempty closed set of real numbers. This paper illustrates this new understanding by extending some discrete results from difference equations to dynamic equations on time scales. Henceforth we consider the nonlinear variable delay dynamic equation

$$x^{\Delta}(t) = F(t, x(\tau(t))), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above, for each fixed $t \in \mathbb{T}$ the function $F(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, and for $x \in \mathbb{R}$ fixed $F(\cdot, x) : \mathbb{T} \to \mathbb{R}$ is right-dense continuous with $F(t, 0) \equiv 0$. Moreover, the variable delay $\tau : \mathbb{T} \to \mathbb{T}$ is nondecreasing with $\tau(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ such that $\lim_{t\to\infty} \tau(t) = \infty$. The initial function associated with (1.1) takes the form $x(t) = \psi(t)$

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for $t \in [\tau(t_0), t_0]$, where ψ is rd-continuous on $[\tau(t_0), t_0]$. Eq. (1.1) is studied extensively by Zhang and Yan in [2] in the case when $\mathbb{T} = \mathbb{Z}$; indeed much of the organization of this paper is motivated by [2]. See also related discussions by Erbe et al. in [3] and by Matsunaga et al. [8]. Another paper on delay dynamic equations is [4]. For more on dynamic equations on time scales, skip ahead to the appendix, Section 5, or consult the recent texts by Bohner and Peterson [5, 6] and Kaymakçalan et al. [7]. To clarify some notation, take $\tau^{-1}(t) := \sup\{s : \tau(s) \le t\}$, $\tau^{-(n+1)}(t) = \tau^{-1}(\tau^{-n}(t))$ for $t \in [\tau(t_0), \infty)_{\mathbb{T}}$, and $\tau^{n+1}(t) = \tau(\tau^n(t))$ for $t \in [\tau^{-3}(t_0), \infty)_{\mathbb{T}}$. By our choice of the delay τ, there exists large $T \in \mathbb{T}$ such that $\tau(t) \ge t_0$ and $\tau^2(t) \le \tau(t) \le t \le \tau^{-1}(\sigma(t))$ for all $t \ge T$. In addition, we always suppose

(H1) there exists continuous $f : \mathbb{R} \to \mathbb{R}$ with $xf(x) > 0$ for $x \ne 0$, and $\alpha, \beta > 0$ such that $-\alpha|x| < f(x) < \beta|x|$, for $x \ne 0$, with

$$f^\dagger(x) := \max\left\{\sup_{0\le u\le|x|} f(u),\ \sup_{0\le u\le|x|}(-f(-u))\right\} \quad x \in \mathbb{R};$$

(H2) there exist rd-continuous functions $a, b : \mathbb{T} \to [0, \infty)$ such that for large $t \in \mathbb{T}$, $x \in \mathbb{R}$, and f from (H1),

$$a(t)\min\{0, f(-x)\} \le F(t, x) \le b(t)\max\{0, f(-x)\};$$

(H3) there exist $A, B > 0$ such that for large $t \in \mathbb{T}$ and a, b from (H2),

$$\int_{\tau(t)}^{\sigma(t)} a(s)\Delta s \le \lambda A \quad \text{and} \quad \int_{\tau(t)}^{\sigma(t)} b(s)\Delta s \le \lambda B,$$

where

$$\lambda := \frac{3}{2} + \frac{1}{2}\frac{\inf\{\mu(t) : t \in \mathbb{T}\}}{\sup\{\tau^{-1}(\sigma(t)) - t : t \in \mathbb{T}\}}. \tag{1.2}$$

It is understood that $\lambda = 3/2$ if either $\inf\{\mu(t)\} = 0$ or $\sup\{\tau^{-1}(\sigma(t)) - t\} = \infty$. For later convenience, define

$$\bar{A} := \max\{1, A\}, \quad \bar{B} := \max\{1, B\}.$$

2. Background lemmas

We will need Lemma 2.1 in the proof of Lemma 2.4.

Lemma 2.1. *For right-dense continuous functions $w : \mathbb{T} \to \mathbb{R}$ and points $c, t \in \mathbb{T}$,*

$$\int_c^t \left(w(s)\int_c^{\sigma(s)} w(z)\Delta z\right)\Delta s = \frac{1}{2}\left(\int_c^t w(s)\Delta s\right)^2 + \frac{1}{2}\int_c^t \mu(s)w^2(s)\Delta s.$$

Proof. Let

$$W(t) := \frac{1}{2}\left(\int_c^t w(s)\Delta s\right)^2 + \frac{1}{2}\int_c^t \mu(s)w^2(s)\Delta s - \int_c^t \left(w(s)\int_c^{\sigma(s)} w(z)\Delta z\right)\Delta s.$$

Then $W(c) = 0$, and

$$W^\Delta(t) = \frac{1}{2}w(t)\left(\int_c^t w(s)\Delta s - \int_c^{\sigma(t)} w(s)\Delta s + \mu(t)w(t)\right) = 0. \quad \square$$

Example 2.2. Let $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n = 0, 1, 2, 3, \ldots\}$. Then for $c, t \in \mathbb{N}_0$ with $c < t$, Lemma 2.1 takes the form

$$\sum_{s=c}^{t-1}\sum_{z=c}^{s}(2s+1)(2z+1)w(s^2)w(z^2) = \frac{1}{2}\left(\sum_{s=c}^{t-1}(2s+1)w(s^2)\right)^2 + \frac{1}{2}\sum_{s=c}^{t-1}(2s+1)^2w^2(s^2).$$

Example 2.3. Let $\mathbb{T} = \{f_n : n \in \mathbb{N}\}$, the Fibonacci numbers, where $f_1 = 1$, $f_2 = 2$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Then for $c, t \in \mathbb{N}$ with $c < t$, Lemma 2.1 takes the form

$$\sum_{n=c}^{t-1}\sum_{s=c}^{n} f_{n-1}f_{s-1}w(f_n)w(f_s) = \frac{1}{2}\left(\sum_{n=c}^{t-1} f_{n-1}w(f_n)\right)^2 + \frac{1}{2}\sum_{n=c}^{t-1}(f_{n-1}w(f_n))^2.$$

Lemma 2.4. *Assume* (H1)–(H3) *hold, with* $\alpha = 1 = \beta$. *Let* x *be a solution of* (1.1); *suppose there exists* $t_1 \in (\tau^{-2}(T), \infty)_\mathbb{T}$ *such that* $\tau^2(t_1) \geq t_0$, (H3) *holds for* $t \geq \tau^2(t_1)$, *and* $x(t_1)x^\sigma(t_1) \leq 0$. *Let* $M > 0$ *be given.*

(i) *If* $x(t) \geq -M$ *for* $t \in [\tau^2(t_1), t_1]_\mathbb{T}$, *then*

$$x(t) \leq B\bar{B}f^\dagger(M) \quad \text{for } t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_\mathbb{T}.$$

(ii) *If* $x(t) \leq M$ *for* $t \in [\tau^2(t_1), t_1]_\mathbb{T}$, *then*

$$x(t) \geq -A\bar{A}f^\dagger(M) \quad \text{for } t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_\mathbb{T}.$$

Proof. We concentrate solely on (i), since the proof of (ii) is similar. Following from $x(t_1)x^\sigma(t_1) \leq 0$, there exists a real number $\xi \in [t_1 - 1, t_1]$ such that

$$x(t_1) + [x^\sigma(t_1) - x(t_1)](\xi - t_1 + 1) = 0. \tag{2.1}$$

Because $f^\dagger$ is nonnegative and nondecreasing, $f(x(t)) \geq -f^\dagger(x(t)) \geq -f^\dagger(M)$ for $t \in [\tau^2(t_1), t_1]_\mathbb{T}$; from (1.1) and (H2) we have

$$x^\Delta(t) \leq b(t)f^\dagger(M), \quad t \in [\tau(t_1), \tau^{-1}(t_1)]_\mathbb{T}, \tag{2.2}$$

so that integration and the fundamental theorem yield

$$x(t_1) - x(\tau(t)) \leq f^\dagger(M)\int_{\tau(t)}^{t_1} b(s)\Delta s, \quad t \in [t_1, \tau^{-1}(t_1)]_\mathbb{T}.$$

Using the characterization of ξ in (2.1), we obtain that for $t \in [t_1, \tau^{-1}(t_1)]_\mathbb{T}$,

$$\begin{aligned} x(\tau(t)) &\geq x(t_1) - f^\dagger(M)\int_{\tau(t)}^{t_1} b(s)\Delta s \\ &= -[x^\sigma(t_1) - x(t_1)](\xi - t_1 + 1) - f^\dagger(M)\int_{\tau(t)}^{t_1} b(s)\Delta s \\ &\geq -f^\dagger(M)\left[(\xi - t_1)\int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} b(s)\Delta s\right], \end{aligned}$$

where we used (2.2) to arrive at the last line. Continuing in this manner, from (H2) and the fact that $f^\dagger(x) < x$ for positive x, we see that

$$
\begin{aligned}
x^\Delta(t) &\leq b(t) f^\dagger \left(f^\dagger(M) \left[(\xi - t_1) \int_{t_1}^{\sigma(t_1)} b(s)\Delta s + \int_{\tau(t)}^{\sigma(t_1)} b(s)\Delta s \right] \right) \\
&\leq B^2 f^\dagger(M) b^\dagger(t) \left[\int_{\tau(t)}^{\sigma(t_1)} b^\dagger(s)\Delta s - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \qquad (2.3)
\end{aligned}
$$

for $t \in [t_1, \tau^{-1}(t_1)]_{\mathbb{T}}$, where $b^\dagger(t) := b(t)/B$. Now by (H3) we know that

$$
0 \leq \zeta := (t_1 - \xi) \int_{t_1}^{\sigma(t_1)} b^\dagger(s)\Delta s + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\Delta s \leq \lambda, \qquad (2.4)
$$

which we consider in the following two cases.

CASE 1: Suppose ζ defined in (2.4) satisfies $\zeta \in (0, 1)$. For $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, we have

$$
\begin{aligned}
x(t) &= x^\sigma(t_1) + \int_{\sigma(t_1)}^{t} x^\Delta(s)\Delta s \\
&\overset{(2.1)}{=} [x^\sigma(t_1) - x(t_1)](t_1 - \xi) + \int_{\sigma(t_1)}^{t} x^\Delta(s)\Delta s \\
&\overset{\text{Theorem 5.4}}{=} (t_1 - \xi)\mu(t_1) x^\Delta(t_1) + \int_{\sigma(t_1)}^{t} x^\Delta(s)\Delta s \\
&\overset{(2.3)}{\leq} (t_1 - \xi)\mu(t_1) B^2 f^\dagger(M) b^\dagger(t_1) \left[\int_{\tau(t_1)}^{\sigma(t_1)} b^\dagger(s)\Delta s - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \\
&\quad + B^2 f^\dagger(M) \int_{\sigma(t_1)}^{t} b^\dagger(s) \left[\int_{\tau(s)}^{\sigma(t_1)} b^\dagger(u)\Delta u - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \Delta s \\
&= B^2 f^\dagger(M) \left\{ (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \left[\int_{\tau(t_1)}^{\sigma(t_1)} b^\dagger(s)\Delta s - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \right. \\
&\quad \left. + \int_{\sigma(t_1)}^{t} b^\dagger(s) \left[\int_{\tau(s)}^{\sigma(t_1)} b^\dagger(u)\Delta u - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \Delta s \right\},
\end{aligned}
$$

where the last equality follows from simple factoring. Continuing,

$$
\begin{aligned}
x(t) &\overset{(H3)}{\leq} B^2 f^\dagger(M) \left\{ (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \left[\lambda - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \right. \\
&\quad \left. + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s) \left[\lambda - \int_{\sigma(t_1)}^{\sigma(s)} b^\dagger(u)\Delta u - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right] \Delta s \right\} \\
&= B^2 f^\dagger(M) \left\{ - \left[(t_1 - \xi)\mu(t_1) b^\dagger(t_1) \right]^2 - (t_1 - \xi)\mu(t_1) b^\dagger(t_1) \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\Delta s \right. \\
&\quad \left. + \lambda\zeta - \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s) \left[\int_{\sigma(t_1)}^{\sigma(s)} b^\dagger(u)\Delta u \right] \Delta s \right\}.
\end{aligned}
$$

Using Lemma 2.1 on the last double integral involving $b^\dagger$,

$$\begin{aligned} x(t) \le\; & B^2 f^\dagger(M)\Bigg\{-\Big[(t_1-\xi)\mu(t_1)b^\dagger(t_1)\Big]^2 - (t_1-\xi)\mu(t_1)b^\dagger(t_1)\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\Delta s \\ & + \lambda\zeta - \frac{1}{2}\left(\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\Delta s\right)^2 - \frac{1}{2}\int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} \mu(s)(b^\dagger(s))^2\Delta s\Bigg\} \\ =\; & B^2 f^\dagger(M)\left(\lambda\zeta - \left[\frac{\zeta^2}{2} + \frac{\big((t_1-\xi)\mu(t_1)b^\dagger(t_1)\big)^2}{2} + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} \frac{\mu(s)}{2}(b^\dagger(s))^2\Delta s\right]\right). \end{aligned}$$

Define

$$m(s) := \begin{cases} (t_1-\xi)\sqrt{\mu(s)}\,b^\dagger(s) & : s \le t_1 \\ \sqrt{\mu(s)}\,b^\dagger(s) & : s > t_1, \end{cases}$$

so that m is right-dense continuous and

$$x(t) \le B^2 f^\dagger(M)\left(\lambda\zeta - \frac{\zeta^2}{2} - \frac{1}{2}\int_{t_1}^{\tau^{-1}(\sigma(t_1))} m^2(s)\Delta s\right).$$

By the Cauchy–Schwarz Inequality [5, Theorem 6.15],

$$\begin{aligned} \int_{t_1}^{\tau^{-1}(\sigma(t_1))} m^2(s)\Delta s \;&\ge\; \frac{1}{\tau^{-1}(\sigma(t_1)) - t_1}\left(\int_{t_1}^{\tau^{-1}(\sigma(t_1))} m(s)\Delta s\right)^2 \\ &=\; \frac{1}{\tau^{-1}(\sigma(t_1)) - t_1}\Bigg((t_1-\xi)(\mu(t_1))^{3/2}b^\dagger(t_1) \\ &\qquad + \int_{\sigma(t_1)}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\sqrt{\mu(s)}\Delta s\Bigg)^2 \\ &\overset{(1.2)}{\ge}\; 2(\lambda - 3/2)\zeta^2. \end{aligned}$$

Thus, for $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$,

$$x(t) \le B^2 f^\dagger(M)\left(\lambda\zeta - \frac{\zeta^2}{2} - (\lambda - 3/2)\zeta^2\right). \tag{2.5}$$

If $q(x) := \lambda x - \frac{x^2}{2} - (\lambda - 3/2)x^2$, then $q'(0) > 0$ and $q'(1) = 2 - \lambda \ge 0$ by the choice of λ in (1.2), so that q is increasing on [0, 1]. Consequently,

$$x(t) \le B^2 f^\dagger(M) \le B\bar{B} f^\dagger(M), \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}.$$

CASE 2: Suppose $1 \le \zeta \le \lambda$ for ζ as in (2.4). Actually, from (H3) and the fact that $b^\dagger = b/B$, we have in this case that $\int_{t_1}^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\Delta s \in [1, \lambda]$. Note that

$$g(t) := \int_t^{\tau^{-1}(\sigma(t_1))} b^\dagger(s)\Delta s - 1, \quad t \in [t_1, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$$

is a delta-differentiable and decreasing function, so that by [5, Theorem 1.16(i)], g is continuous on $t \in [t_1, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$. Since $g(t_1) \geq 0$ and $g(\tau^{-1}(\sigma(t_1))) = -1 < 0$, by the Intermediate Value Theorem [5, Theorem 1.115], there exists $t_2 \in [t_1, \tau^{-1}(\sigma(t_1)))_{\mathbb{T}}$ such that either $g(t_2) = 0$ or $g(t_2) > 0 > g^{\sigma}(t_2)$. Either way,

$$\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\Delta s < 1 \leq \int_{t_2}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\Delta s = \mu(t_2)b^{\dagger}(t_2) + \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\Delta s,$$

ergo there exists a real number $\phi \in [t_2 - 1, t_2)$ such that

$$\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\Delta s + (t_2 - \phi)\mu(t_2)b^{\dagger}(t_2) = 1. \tag{2.6}$$

Using (2.1) and (2.2), and the definitions of $b^{\dagger}$ and $\bar{B}$, we have for $t \in [t_1, t_2]_{\mathbb{T}}$ that

$$\begin{aligned} x(t) &= (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\ &\leq (t_1 - \xi)\mu(t_1)b(t_1)f^{\dagger}(M) + \int_{\sigma(t_1)}^{t} b(s)f^{\dagger}(M)\Delta s \\ &\leq Bf^{\dagger}(M)\left[(t_1 - \xi)\mu(t_1)b^{\dagger}(t_1) + \int_{\sigma(t_1)}^{t_2} b^{\dagger}(s)\Delta s\right] \\ &\leq Bf^{\dagger}(M)\int_{t_1}^{t_2} b^{\dagger}(s)\Delta s < B\bar{B}f^{\dagger}(M), \end{aligned}$$

where the last inequality follows from our choice of t_2. For $t \in [t_2, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, with (2.1) we see that

$$\begin{aligned} x(t) &= (t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + \int_{\sigma(t_1)}^{t} x^{\Delta}(s)\Delta s \\ &= \left[(t_1 - \xi)\mu(t_1)x^{\Delta}(t_1) + (\phi - t_2 + 1)\mu(t_2)x^{\Delta}(t_2) + \int_{\sigma(t_1)}^{t_2} x^{\Delta}(s)\Delta s\right] \\ &\quad + \left[(t_2 - \phi)\mu(t_2)x^{\Delta}(t_2) + \int_{\sigma(t_2)}^{t} x^{\Delta}(s)\Delta s\right] = S_1 + S_2, \end{aligned}$$

where S_1 is the first grouping and S_2 the second. Using (2.2) for S_1 and (2.3) for S_2,

$$S_1 \leq Bf^{\dagger}(M)\left[(t_1 - \xi)\mu(t_1)b^{\dagger}(t_1) + (\phi - t_2)\mu(t_2)b^{\dagger}(t_2) + \int_{\sigma(t_1)}^{\sigma(t_2)} b^{\dagger}(s)\Delta s\right]$$

and

$$\begin{aligned} S_2 &\leq B^2 f^{\dagger}(M)(t_2 - \phi)\mu(t_2)b^{\dagger}(t_2)\left[\int_{\tau(t_2)}^{\sigma(t_1)} b^{\dagger}(s)\Delta s - (t_1 - \xi)\mu(t_1)b^{\dagger}(t_1)\right] \\ &\quad + B^2 f^{\dagger}(M)\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\left[\int_{\tau(s)}^{\sigma(t_1)} b^{\dagger}(u)\Delta u - (t_1 - \xi)\mu(t_1)b^{\dagger}(t_1)\right]\Delta s. \end{aligned}$$

Then continuing for $t \in [t_2, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$ while recalling B, $B^2 \le B\bar{B}$ and (2.6),

$$\begin{aligned} x(t) \le B\bar{B} f^{\dagger}(M) \Bigg(& \left[(t_1-\xi)\mu(t_1)b^{\dagger}(t_1) + (\phi - t_2)\mu(t_2)b^{\dagger}(t_2) + \int_{\sigma(t_1)}^{\sigma(t_2)} b^{\dagger}(s)\Delta s \right] \\ & \times \left[\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\Delta s + (t_2-\phi)\mu(t_2)b^{\dagger}(t_2) \right] \\ & + (t_2-\phi)\mu(t_2)b^{\dagger}(t_2) \left[\int_{\tau(t_2)}^{\sigma(t_1)} b^{\dagger}(s)\Delta s - (t_1-\xi)\mu(t_1)b^{\dagger}(t_1) \right] \\ & + \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s) \left[\int_{\tau(s)}^{\sigma(t_1)} b^{\dagger}(u)\Delta u - (t_1-\xi)\mu(t_1)b^{\dagger}(t_1) \right] \Delta s \Bigg). \end{aligned}$$

Proceeding by rearranging,

$$\begin{aligned} x(t) \le\ & B\bar{B} f^{\dagger}(M) \Bigg(\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s) \left[(\phi-t_2)\mu(t_2)b^{\dagger}(t_2) + \int_{\tau(s)}^{\sigma(t_2)} b^{\dagger}(u)\Delta u \right] \Delta s \\ & + (t_2-\phi)\mu(t_2)b^{\dagger}(t_2) \left[(\phi-t_2)\mu(t_2)b^{\dagger}(t_2) + \int_{\tau(t_2)}^{\sigma(t_2)} b^{\dagger}(s)\Delta s \right] \Bigg) \\ \overset{\text{(H3)}}{\le}\ & B\bar{B} f^{\dagger}(M) \Bigg(\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s) \left[(\phi-t_2)\mu(t_2)b^{\dagger}(t_2) + \lambda - \int_{\sigma(t_2)}^{\sigma(s)} b^{\dagger}(u)\Delta u \right] \Delta s \\ & + (t_2-\phi)\mu(t_2)b^{\dagger}(t_2) \left[(\phi-t_2)\mu(t_2)b^{\dagger}(t_2) + \lambda \right] \Bigg) \\ \overset{(2.6)}{=}\ & B\bar{B} f^{\dagger}(M) \Bigg(\lambda - \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s) \int_{\sigma(t_2)}^{\sigma(s)} b^{\dagger}(u)\Delta u \Delta s - [(t_2-\phi)\mu(t_2)b^{\dagger}(t_2)]^2 \\ & - (t_2-\phi)\mu(t_2)b^{\dagger}(t_2) \int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} b^{\dagger}(s)\Delta s \Bigg) \\ =\ & B\bar{B} f^{\dagger}(M) \left(\lambda - \frac{1}{2} - \frac{1}{2}\int_{\sigma(t_2)}^{\tau^{-1}(\sigma(t_1))} \mu(s)\left(b^{\dagger}(s)\right)^2 \Delta s - \frac{1}{2}\left[(t_2-\phi)\mu(t_2)b^{\dagger}(t_2)\right]^2 \right) \end{aligned}$$

using Lemma 2.1 and (2.6) again. Thus, as in (2.5),

$$x(t) \le B\bar{B} f^{\dagger}(M) \left(\lambda - \frac{1}{2} - (\lambda - 3/2) \right) = B\bar{B} f^{\dagger}(M), \quad t \in [t_2, \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}. \quad \square$$

Lemma 2.5. *Suppose that* (H1)–(H3) *hold with* $\alpha = 1 = \beta$, *and that* $A\bar{A}B\bar{B} \le 1$. *Let* x *be a solution of* (1.1) *and* $t_1 \in T$ *be as in Lemma 2.4. Then for any* $M > 0$ *the following hold.*

(i) *If* $B \le 1$ *and* $-A\bar{A}M \le x(t) \le M$ *for* $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, *then*

$$-A\bar{A} f^{\dagger}(M) \le x(t) \le f^{\dagger}(M), \quad t \in [\sigma(t_1), \infty)_{\mathbb{T}}.$$

(ii) *If* $A \le 1$ *and* $-M \le x(t) \le B\bar{B}M$ *for* $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$, *then*

$$-f^{\dagger}(M) \le x(t) \le B\bar{B} f^{\dagger}(M), \quad t \in [\sigma(t_1), \infty)_{\mathbb{T}}.$$

Proof. The proof of (ii) is similar to that for (i) and is omitted. Thus, suppose that $B \leq 1$ and $-A\bar{A}M \leq x(t) \leq M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$. Since B and $A\bar{A}B\bar{B} \leq 1$ and $\alpha = 1 = \beta$, (H1) and (H3) imply

$$x(t) \geq B\bar{B}x(t) \geq -A\bar{A}B\bar{B}M \geq -M, \quad t \in [\tau^2(t_1), t_1]_{\mathbb{T}}.$$

Then by Lemma 2.4(i), $x(t) \leq B\bar{B}f^{\dagger}(M) \leq f^{\dagger}(M)$ for $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$; by Lemma 2.4(ii), $x(t) \geq -A\bar{A}f^{\dagger}(M)$ for $t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}$, so that

$$-A\bar{A}f^{\dagger}(M) \leq x(t) \leq f^{\dagger}(M), \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}}.$$

Apply this reasoning recursively across overlapping intervals: $-A\bar{A}M \leq x(t) \leq M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$ implies

$$-A\bar{A}M \leq -A\bar{A}f^{\dagger}(M) \leq x(t) \leq f^{\dagger}(M) \leq M, \quad t \in [\sigma(t_1), \tau^{-1}(\sigma(t_1))]_{\mathbb{T}},$$

so that ultimately we have, for $t_2 := \tau^{-1}(\sigma(t_1))$,

$$-A\bar{A}M \leq x(t) \leq M, \quad t \in [\tau^2(t_2), t_2]_{\mathbb{T}},$$

which leads to

$$-A\bar{A}f^{\dagger}(M) \leq x(t) \leq f^{\dagger}(M), \quad t \in [\sigma(t_2), \tau^{-1}(\sigma(t_2))]_{\mathbb{T}}.$$

Under the blanket assumption that $\lim_{t\to\infty} \tau(t) = \infty$, we get the result. □

3. Solutions of (1.1) go to zero

We now state our main result on the asymptotic behavior of solutions of (1.1) under various assumptions, namely that the trivial solution of (1.1) is asymptotically stable.

Theorem 3.1. *Suppose that* (H1)–(H3) *hold with* $\alpha = 1 = \beta$, *and that* $A\bar{A}B\bar{B} \leq 1$. *If there exists* $t^* \in \mathbb{T}$ *and a collection of continuous functions* $\{H(t,\cdot) : (0,\infty) \to [0,\infty)\}_{t\in\mathbb{T}}$ *such that, for any* $\epsilon > 0$ *and* $t \in [t^*, \infty)_{\mathbb{T}}$,

$$\sup_{x\geq\epsilon} F(t,x) \leq -H(t,\epsilon), \quad \inf_{x\leq-\epsilon} F(t,x) \geq H(t,\epsilon), \quad \int_{t^*}^{\infty} H(t,\epsilon)\Delta t = \infty, \tag{3.1}$$

then every solution of (1.1) *goes to zero in the limit. In other words, the trivial solution of* (1.1) *is asymptotically stable.*

Proof. Let x be a solution of (1.1). If x is eventually negative or eventually positive, then x must be eventually nondecreasing or eventually nonincreasing by (H2). Without loss of generality suppose x is eventually positive, and thus eventually nonincreasing. If $x(t) \to \varepsilon > 0$, then $\sup_{x\geq\varepsilon} F(t,x) = 0$ for all $t \geq t^*$ by (1.1), so $0 \leq -H(t,\varepsilon) \leq 0$ for all $t \geq t^*$ by (3.1); in other words, $H \equiv 0$ on $[t^*,\infty)_{\mathbb{T}}$, a contradiction of $\int_{t^*}^{\infty} H(t,\epsilon)\Delta t = \infty$. Therefore any solution x that is eventually of one sign goes to zero in the limit. Next assume that x is an oscillatory solution of (1.1). Then x has a generalized zero in any neighborhood of infinity, so there exists an increasing sequence of points $\{t_n \in \mathbb{T}\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$ and $x(t_n)x^{\sigma}(t_n) \leq 0$. As $A\bar{A}B\bar{B} \leq 1$, either $A \leq 1$ or $B \leq 1$. If the latter, let $M > 0$ be given such that $-A\bar{A}M \leq x(t) \leq M$ for $t \in [\tau^2(t_1), t_1]_{\mathbb{T}}$. By Lemma 2.5(i), $-A\bar{A}f^{\dagger}(M) \leq x(t) \leq f^{\dagger}(M)$ for $t \in [\sigma(t_1), \infty)_{\mathbb{T}}$. Let $f_1^{\dagger} := f^{\dagger}$ and $f_{n+1}^{\dagger} := f^{\dagger} \circ f_n^{\dagger}$ for $n \in \mathbb{N}$. By successive use of the above argument, we deduce that

$$-A\bar{A}f_n^{\dagger}(M) \leq x(t) \leq f_n^{\dagger}(M), \quad t \in [\sigma(t_n), \infty)_{\mathbb{T}}. \tag{3.2}$$

Since $\alpha = 1 = \beta$, assumption (H1) implies that the sequence $\{f_n^\dagger(M)\}_{n\in\mathbb{N}}$ is nonincreasing. If $M_0 := \lim_{n\to\infty} f_n^\dagger(M)$, then $M_0 = f^\dagger(M_0)$. If $M_0 > 0$, then since $f(M_0) < M_0$ by (H1), set $f(M_0) = M_0 - \varepsilon$. By the continuity of f, there exists $\delta > 0$ such that $|f(x) - f(M_0)| < \varepsilon/2$ for all $x \in (M_0 - \delta, M_0]$. Therefore $|f(x) - M_0 + \varepsilon| < \varepsilon/2$, which means $M_0 - 3\varepsilon/2 < f(x) < M_0 - \varepsilon/2$ for all $x \in (M_0 - \delta, M_0]$. But $f(x) < x$ on $[0, m_0 - \delta]$ implies that $f(x) < M_0 - \delta < M_0$ on $[0, M_0 - \delta]$. Consequently f is bounded away from M_0 on $[0, M_0]$, a contradiction of $M_0 = f^\dagger(M_0)$, so that $M_0 = 0$. From (3.2) we have that $\lim_{t\to\infty} x(t) = 0$. In a similar way we arrive at the identical conclusion if $A \le 1$. $\square$

In the following corollary, we consider the related delay dynamic equation

$$x^{\Delta}(t) + p(t)g(x(\tau(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.3}$$

where $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous and $g : \mathbb{R} \to \mathbb{R}$ is continuous with $xg(x) > 0$ for $x \neq 0$.

Corollary 3.2. *Let $\alpha_*, \beta_* > 0$ be such that*

$$-\alpha_*|x| \le g(x) \le \beta_*|x|, \quad x \neq 0.$$

Assume for all large $t \in \mathbb{T}$ that

$$(\alpha_*\beta_*^2)^{1/3} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \le \lambda \quad \text{if } \alpha_* \le \beta_*, \quad (\alpha_*^2\beta_*)^{1/3} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \le \lambda \quad \text{if } \alpha_* > \beta_*. \tag{3.4}$$

If $\int_{t_0}^{\infty} p(t)\Delta t = \infty$, then every solution of (3.3) goes to zero in the limit.

Proof. Take $F(t, x) := -p(t)g(x)$, $a(t) := \alpha_* p(t)$, $b(t) := \beta_* p(t)$,

$$f(x) := \begin{cases} g(x)/\beta_* & : x \ge 0 \\ g(x)/\alpha_* & : x < 0, \end{cases}$$

and $H(t, \epsilon) := p(t) \min\left\{\inf_{x\ge\epsilon} g(x), -\sup_{x\le-\epsilon} g(x)\right\}$. Then (H1) and (H2) are satisfied with $\alpha = 1 = \beta$, as are the conditions in (3.1). Suppose $\alpha_* \le \beta_*$; the other case is similar and is omitted. If $A := (\alpha_*/\beta_*)^{2/3}$ and $B := (\beta_*/\alpha_*)^{1/3}$, then by (3.4) we have that (H3) holds. Moreover, we see that

$$A\bar{A}B\bar{B} = AB^2 = \left(\frac{\alpha_*}{\beta_*}\right)^{2/3} \left(\frac{\beta_*}{\alpha_*}\right)^{2/3} = 1.$$

By Theorem 3.1, every solution of (3.3) goes to zero in the limit. $\square$

4. Logistic delay equation on isolated time scales

Let $\mathbb{T}$ be a time scale unbounded above, with every point both left and right scattered. Consider the logistic dynamic equation [6, (2.28)],

$$y^{\Delta} = \frac{py(1 - y/N)}{1 + \frac{\mu p}{N}y} \quad \text{or} \quad y^{\sigma} = \frac{y(1 + \mu p)}{1 + \frac{\mu p}{N}y}, \quad \frac{py}{N} \in \mathcal{R},$$

with $\mathcal{R}$ given in Definition 5.5. Because of the isolated nature of each point in $\mathbb{T}$ in this section, these two forms are equivalent by Theorem 5.4 in the appendix. We introduce a delay $\tau : \mathbb{T} \to \mathbb{T}$

and concentrate on

$$y^{\sigma}(t) = \frac{y(t)(1+\mu(t)p(t))}{1+\frac{\mu(t)p(t)}{N}y(\tau(t))}, \quad t \in \mathbb{T}, \tag{4.1}$$

assuming $N > 0$ is fixed, $p > 0$ on $\mathbb{T}$, and $p(y \circ \tau)/N \in \mathcal{R}$. We would like to show conditions under which positive solutions of (4.1) go to the "carrying capacity" N. In (4.1), let $y = N\mathrm{e}^{x}$, $y^{\sigma} = N\mathrm{e}^{x\circ\sigma}$, and $y \circ \tau = N\mathrm{e}^{x\circ\tau}$ to obtain

$$x^{\Delta}(t) = \frac{1}{\mu(t)} \ln\left\{\frac{1+\mu(t)p(t)}{1+\mu(t)p(t)\mathrm{e}^{x(\tau(t))}}\right\}. \tag{4.2}$$

If

$$F(t,x) := \frac{1}{\mu(t)} \ln\left\{\frac{1+\mu(t)p(t)}{1+\mu(t)p(t)\mathrm{e}^{x}}\right\},$$

then $F(t,\cdot)$ is continuous with $F(t,0) \equiv 0$. For fixed $t \in \mathbb{T}$ and $x > 0$, $F(t,x) \geq -x/\mu(t)$, so we take $a(t) := 1/\mu(t)$; for $x < 0$, $F(t,x) \leq -xp(t)/(1+\mu(t)p(t))$ implies we should have $b(t) := p(t)/(1+\mu(t)p(t))$. A direct consequence of these choices is that conditions (H1) and (H2) are met with $\alpha = 1 = \beta$. Finally, notice that $-F(t,\epsilon) \geq F(t,-\epsilon)$ for $t \in \mathbb{T}$ and $\epsilon > 0$, so that

$$\sup_{x\geq\epsilon} F(t,x) = F(t,\epsilon) \leq -F(t,-\epsilon), \qquad \inf_{x\leq-\epsilon} F(t,x) = F(t,-\epsilon)$$

leads us to define $H(t,\epsilon) := F(t,-\epsilon)$.

Example 4.1. Let $\mathbb{T} = h\mathbb{Z}$ for some $h \in (0,1)$, $\tau(t) := t - hk$ for $t \in \mathbb{T}$ and $k \in \mathbb{N}$, and $p(t) \in (0,\bar{p}]$ for all $t \in \mathbb{T}$ for some $\bar{p} > 0$, such that $\lim_{t\to\infty} p(t) \neq 0$. If

$$\bar{p} \in \left(0, \frac{\lambda^3}{h[(k+1)^3 - \lambda^3]}\right] \quad \text{for } \lambda = \frac{3k+4}{2(k+1)},$$

then every positive solution of (4.1) goes to N in the limit.

Proof. Note that $\mu(t) \equiv h$. Take $A := \frac{k+1}{\lambda}$ and $B := \frac{h\bar{p}(k+1)}{\lambda(1+h\bar{p})}$. Then

$$\int_{\tau(t)}^{\sigma(t)} a(s)\Delta s = \int_{t-hk}^{t+h} \frac{1}{h}\Delta s = k+1 = \lambda A,$$

$$\int_{\tau(t)}^{\sigma(t)} b(s)\Delta s = \int_{t-hk}^{t+h} \frac{p(s)}{1+hp(s)}\Delta s \leq \int_{t-hk}^{t+h} \frac{\bar{p}}{1+h\bar{p}}\Delta s = \frac{(k+1)h\bar{p}}{1+h\bar{p}} = \lambda B,$$

$$\text{and} \quad \int_{h}^{\infty} F(t,-\epsilon)\Delta t = \infty.$$

Moreover, if $\bar{p} \in \left(0, \frac{\lambda^3}{h[(k+1)^3-\lambda^3]}\right]$, then

$$A\bar{A}B\bar{B} = AAB(1) = \frac{h\bar{p}(k+1)^3}{\lambda^3(1+h\bar{p})} \leq 1.$$

Hence by Theorem 3.1, every solution x of (4.2) goes to zero in the limit. But then every positive solution $y = N\mathrm{e}^{x}$ of (4.1) goes to N. $\square$

Remark. Using the alternative substitutions

$$z(t) = \frac{\mu(\tau^{-1}(t))p(\tau^{-1}(t))}{N} y(t), \quad \ell(t) = (1 + \mu(t)p(t)) \frac{\mu^\sigma(\tau^{-1}(t))p^\sigma(\tau^{-1}(t))}{\mu(\tau^{-1}(t))p(\tau^{-1}(t))},$$

Eq. (4.1) could also be rewritten as

$$z^\sigma(t) = \frac{\ell(t)z(t)}{1 + z(\tau(t))}.$$

5. Appendix on time scales

The definitions below merely serve as a preliminary introduction to the time-scale calculus; they can be found in the context of a much more robust treatment than is allowed here in the texts [5] and [6] and the references therein.

Definition 5.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}) \quad \text{for all } t \in \mathbb{T}.$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. Define the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ is unbounded above and has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ and define the time-scale interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$ the point t is right-scattered, while if $\rho(t) < t$ then t is left-scattered. If $\sigma(t) = t$ the point t is right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then t is left-dense.

Definition 5.2. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t such that, for all $s \in U$,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|.$$

Call $y^\Delta(t)$ the (delta) derivative of y at t.

Definition 5.3. If $F^\Delta(t) = f(t)$ then define the (Cauchy) delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

The following theorem is due to Hilger [1].

Theorem 5.4. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(1) *If f is differentiable at t, then f is continuous at t.*
(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(3) *If f is differentiable and t is right-dense, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(4) *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.*

Next we define the important concept of right-dense continuity. An important fact concerning right-dense continuity is that every right-dense continuous function has a delta antiderivative [5, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists.

Definition 5.5. A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (denoted as $f \in C_{rd}(\mathbb{T}; \mathbb{R})$) provided f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

A function p is regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}$, and

$$\mathcal{R} := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) : 1 + \mu(t) p(t) \neq 0, t \in \mathbb{T}\}.$$

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