

Second-Order n -point Problems on Time Scales with Changing-Sign Nonlinearity

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Abstract

In this study, conditions for the existence of at least one positive solution to a non-linear second-order multipoint eigenvalue problem on time scales are discussed. Here the nonlinearity is allowed to take on negative values. The results extend previous work on the continuous case, and are based on the Guo–Krasnoselskii fixed point theorem.

AMS subject classification: 34B05, 39A10.

Keywords: Time scales, positive solutions, cone, fixed point theorem.

1. Introduction

We are interested in the second-order n -point time scale boundary value problem

$$(py^\nabla)^\Delta(t) - q(t)y(t) + \lambda f(t, y(t)) = 0, \quad t \in (t_1, t_n)_\mathbb{T}, \quad (1.1)$$

$$\alpha y(t_1) - \beta p(t_1)y^\nabla(t_1) = \sum_{i=2}^{n-1} a_i y(t_i), \quad \gamma y(t_n) + \delta p(t_n)y^\nabla(t_n) = \sum_{i=2}^{n-1} b_i y(t_i), \quad (1.2)$$

where $n \geq 3$ and

$$p, q : [t_1, t_n]_\mathbb{T} \rightarrow (0, \infty), \quad p \in C^\Delta[t_1, t_n]_\mathbb{T}, \quad q \in C[t_1, t_n]_\mathbb{T}; \quad (1.3)$$

the points $t_i \in \mathbb{T}_\kappa$ for $i \in \{1, 2, \dots, n\}$ with $t_1 < t_2 < \dots < t_n$; the real scalar $\lambda \in (0, \infty)$;

$$\alpha, \gamma \in [0, \infty), \quad \beta, \delta \in (0, \infty), \quad \alpha\gamma + \alpha\delta + \beta\gamma > 0, \\ a_i, b_i \in [0, \infty), \quad i \in \{2, \dots, n-1\}; \quad (1.4)$$

the continuous function $f : (t_1, t_n)_{\mathbb{T}} \times [0, \infty) \rightarrow (-\infty, \infty)$ is such that the following hold:

$$\lim_{y \rightarrow +\infty} \frac{f(t, y)}{y} \stackrel{\text{unif}}{=} +\infty, \quad t \in [t_2, t_3]_{\mathbb{T}}; \quad -u(t) \leq f(t, y) \leq z(t)h(y) \quad (1.5)$$

for right-dense continuous functions $u, z : (t_1, t_n)_{\mathbb{T}} \rightarrow (0, \infty)$ and continuous function $h : [0, \infty) \rightarrow (0, \infty)$. Problem (1.1), (1.2) is a generalization to time scales of the problem when \mathbb{T} is restricted to \mathbb{R} on the unit interval in Zhang and Liu [19], which extends the discussion found in Ma and Thompson [15]. Therefore the results here generalize and extend those works to the discrete and quantum calculus in particular, as well as to arbitrary time scales. See also related time-scale boundary value problems found in Anderson [1,2], Atici and Guseinov [4], Kaufmann [12], Kaufmann and Raffoul [13], Kong and Kong [14], Peterson, Raffoul, and Tisdell [16], and Sun and Li [17, 18]. Recent papers on singular problems on time scales include Bohner and Luo [6] and DaCunha, Davis, and Singh [9]. For more general information concerning dynamic equations on time scales, introduced by Aulbach and Hilger [5] and Hilger [11], see the excellent text by Bohner and Peterson [7] and the follow-up text [8].

2. Time Scale Primer

Any arbitrary nonempty closed subset of the reals \mathbb{R} can serve as a time scale \mathbb{T} ; see [7, 8]. For $t \in \mathbb{T}$ define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. The graininess operators $\mu_\sigma, \mu_\rho : \mathbb{T} \rightarrow [0, \infty)$ are defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at all right dense points of \mathbb{T} and its left sided limit exists (is finite) at left dense points of \mathbb{T} . The set of all right dense continuous functions on \mathbb{T} is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Define the set \mathbb{T}_κ by $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$ if \mathbb{T} has a right scattered minimum m and $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. In a similar vein, $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$ if \mathbb{T} has a left scattered maximum M and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. We take $\mathbb{T}_\kappa^\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

Definition 2.1. [Delta Derivative] Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

The function $f^\Delta(t)$ is the delta derivative of f at t .

Definition 2.2. [Nabla Derivative] For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \epsilon|\rho(t) - s| \quad \text{for all } s \in U.$$

The function $f^\nabla(t)$ is the nabla derivative of f at t .

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$.

Definition 2.3. [Delta Integral] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \rightarrow \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$, then F is a delta antiderivative of f . In this case the integral is given by the formula

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

All right-dense continuous functions are delta integrable; see [7, Theorem 1.74].

3. Linear Preliminaries

We first construct Green's function for the second-order boundary value problem

$$(py^\nabla)^\Delta(t) - q(t)y(t) + \lambda u(t) = 0, \quad t_1 < t < t_n, \quad (3.1)$$

$$\alpha y(t_1) - \beta p(t_1)y^\nabla(t_1) = 0, \quad \gamma y(t_n) + \delta p(t_n)y^\nabla(t_n) = 0, \quad (3.2)$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, and u is as in (1.5). The techniques here are similar to those found in [3, 4] for time scales, and in [15, 19] for the continuous case.

Denote by ϕ and ψ the solutions of the corresponding homogeneous equation

$$(py^\nabla)^\Delta(t) - q(t)y(t) = 0, \quad t \in [t_1, t_n), \quad (3.3)$$

under the initial conditions

$$\psi(t_1) = \beta, p(t_1)\psi^\nabla(t_1) = \alpha, \quad (3.4)$$

$$\phi(t_n) = \delta, p(t_n)\phi^\nabla(t_n) = -\gamma, \quad (3.5)$$

so that ψ and ϕ satisfy the first and second boundary conditions in (3.2), respectively. Set

$$d = -W_t(\psi, \phi) = p(t)\psi^\nabla(t)\phi(t) - \psi(t)p(t)\phi^\nabla(t). \quad (3.6)$$

Since the Wronskian of any two solutions is independent of t , evaluating at $t = t_1$, $t = t_n$, and using the boundary conditions (3.4), (3.5) yields

$$d = \alpha\phi(t_1) - \beta p(t_1)\phi^\nabla(t_1) = \gamma\psi(t_n) + \delta p(t_n)\psi^\nabla(t_n).$$

In addition $d \neq 0$ if and only if the homogeneous equation (3.3) has only the trivial solution satisfying the boundary conditions (3.2).

Lemma 3.1. [4, Theorem 4.2] Assume (1.3), (1.4), and (1.5). If $d \neq 0$, then the nonhomogeneous boundary value problem (3.1), (3.2), has a unique solution y for which the formula

$$y(t) = \lambda \int_{t_1}^{t_n} G(t, s)u(s)\Delta s, \quad t \in [\rho(t_1), t_n]$$

holds, where the function $G(t, s)$ is given by

$$G(t, s) = \frac{1}{d} \begin{cases} \psi(t)\phi(s), & \rho(t_1) \leq t \leq s \leq t_n \\ \psi(s)\phi(t), & \rho(t_1) \leq s \leq t \leq t_n, \end{cases} \quad (3.7)$$

and $G(t, s)$ is Green's function of the boundary value problem (3.1), (3.2).

Lemma 3.2. [4, Lemma 5.1] Assume (1.3) and (1.4). Then the functions ψ and ϕ satisfy

$$\begin{aligned} \psi(t) > 0, \quad t \in [\rho(t_1), t_n], & \quad p(t)\psi^\nabla(t) \geq 0, \quad t \in (\rho(t_1), t_n], \\ \phi(t) > 0, \quad t \in [\rho(t_1), t_n], & \quad p(t)\phi^\nabla(t) \leq 0, \quad t \in (\rho(t_1), t_n]. \end{aligned}$$

Lemma 3.3. Let (1.3) and (1.4) hold. For any $s, t \in [\rho(t_1), t_n]_{\mathbb{T}}$, Green's function satisfies

$$0 \leq \Gamma G(s, s) \leq G(t, s) \leq G(s, s), \quad (3.8)$$

where

$$\Gamma := \min \left\{ \frac{\beta}{\psi(t_n)}, \frac{\delta}{\phi(\rho(t_1))} \right\} < 1. \quad (3.9)$$

Proof. Recall from (1.4) that $\beta, \delta > 0$. By (3.7) and the previous lemma,

$$1 \geq \frac{G(t, s)}{G(s, s)} = \begin{cases} \frac{\psi(t)}{\psi(s)}, & \rho(t_1) \leq t \leq s \leq t_n, \\ \frac{\phi(s)}{\phi(t)}, & \rho(t_1) \leq s \leq t \leq t_n, \end{cases} \geq \begin{cases} \frac{\beta}{\psi(t_n)}, & \rho(t_1) \leq t \leq s \leq t_n, \\ \frac{\delta}{\phi(\rho(t_1))}, & \rho(t_1) \leq s \leq t \leq t_n. \end{cases}$$

For the remainder of the paper set

$$D := \begin{vmatrix} -\sum_{i=2}^{n-1} a_i \psi(t_i) & d - \sum_{i=2}^{n-1} a_i \phi(t_i) \\ d - \sum_{i=2}^{n-1} b_i \psi(t_i) & -\sum_{i=2}^{n-1} b_i \phi(t_i) \end{vmatrix}. \quad (3.10)$$

■

Lemma 3.4. [3, Lemma 2.3] Assume (1.3), (1.4), and (1.5). If

$$D \neq 0 \quad \text{and} \quad \int_{t_1}^{t_n} G(s, s)u(s)\Delta s < \infty,$$

then the nonhomogeneous dynamic equation (3.1) with boundary conditions (1.2) has a unique solution w for which the formula

$$w(t) = \lambda \left(\int_{t_1}^{t_n} G(t, s)u(s)\Delta s + A(u)\psi(t) + B(u)\phi(t) \right), \quad t \in [\rho(t_1), t_n] \quad (3.11)$$

holds, where the function $G(t, s)$ is Green's function (3.7) of the boundary value problem (3.1), (3.2), and the functionals A and B are defined by

$$A(u) := \frac{1}{D} \begin{vmatrix} \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i, s) u(s) \Delta s & d - \sum_{i=2}^{n-1} a_i \phi(t_i) \\ \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i, s) u(s) \Delta s & - \sum_{i=2}^{n-1} b_i \phi(t_i) \end{vmatrix}, \quad (3.12)$$

$$B(u) := \frac{1}{D} \begin{vmatrix} - \sum_{i=2}^{n-1} a_i \psi(t_i) & \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i, s) u(s) \Delta s \\ d - \sum_{i=2}^{n-1} b_i \psi(t_i) & \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i, s) u(s) \Delta s \end{vmatrix}. \quad (3.13)$$

Lemma 3.5. Let (1.3), (1.4), and (1.5) hold, and assume

$$D < 0, \quad d - \sum_{i=2}^{n-1} a_i \phi(t_i) > 0, \quad d - \sum_{i=2}^{n-1} b_i \psi(t_i) > 0 \quad (3.14)$$

for d and D given in (3.6) and (3.10), respectively. Then the unique solution w as in (3.11) of the problem (3.1), (1.2) satisfies

$$\Gamma \|w\| \leq w(t) \leq \lambda \Gamma \xi, \quad t \in [\rho(t_1), t_n], \quad \|w\| := \max_{t \in [\rho(t_1), t_n]} w(t),$$

where Γ is given in (3.9), and

$$\xi := \frac{1}{\Gamma} (1 + A\psi(t_n) + B\phi(\rho(t_1))) \int_{t_1}^{t_n} G(s, s) u(s) \Delta s \quad (3.15)$$

for

$$A := \frac{1}{D} \begin{vmatrix} \sum_{i=2}^{n-1} a_i & d - \sum_{i=2}^{n-1} a_i \phi(t_i) \\ \sum_{i=2}^{n-1} b_i & - \sum_{i=2}^{n-1} b_i \phi(t_i) \end{vmatrix}, \quad B := \frac{1}{D} \begin{vmatrix} - \sum_{i=2}^{n-1} a_i \psi(t_i) & \sum_{i=2}^{n-1} a_i \\ d - \sum_{i=2}^{n-1} b_i \psi(t_i) & \sum_{i=2}^{n-1} b_i \end{vmatrix}.$$

Proof. From Lemma 3.3, Green's function (3.7) satisfies $0 \leq G(t, s) \leq G(s, s)$ for $t \in [\rho(t_1), t_n]$, so that for all $t \in [\rho(t_1), t_n]$,

$$\begin{aligned} w(t) &\leq \lambda \left(\int_{t_1}^{t_n} G(s, s) u(s) \Delta s + A\psi(t) \int_{t_1}^{t_n} G(s, s) u(s) \Delta s \right. \\ &\quad \left. + B\phi(t) \int_{t_1}^{t_n} G(s, s) u(s) \Delta s \right) \\ &\leq \lambda (1 + A\psi(t_n) + B\phi(\rho(t_1))) \int_{t_1}^{t_n} G(s, s) u(s) \Delta s = \lambda \Gamma \xi \end{aligned}$$

for Γ as in (3.9) and ξ as in (3.15). For all $t \in [\rho(t_1), t_n]$,

$$\begin{aligned} w(t) &= \lambda \left(\int_{t_1}^{t_n} \frac{G(t, s)}{G(s, s)} G(s, s) u(s) \Delta s + A(u) \psi(t) + B(u) \phi(t) \right) \\ &\geq \lambda \left(\int_{t_1}^{t_n} \Gamma G(s, s) u(s) \Delta s + \beta A(u) + \delta B(u) \right) \\ &\geq \Gamma \lambda \left(\int_{t_1}^{t_n} G(s, s) u(s) \Delta s + A(u) \psi(t_n) + B(u) \phi(\rho(t_1)) \right) \\ &\geq \Gamma \|w\|. \end{aligned}$$

■

Remark 3.6. Suppose (3.14) does not hold. For example, let $n = 3$, $p(t) \equiv 1 = \alpha = \gamma$, $q(t) \equiv 0 = \beta = \delta = a_2$, and $t_1 = 0$. Then (3.1), (1.2) becomes

$$y^{\nabla\Delta}(t) + u(t) = 0, \quad t_1 < t < t_3, \quad y(t_1) = 0, \quad y(t_3) = b_2 y(t_2).$$

Note that $\psi(t) = t$, $d = t_3$, and $D = t_3(b_2 t_2 - t_3)$. If $D > 0$, then $b_2 t_2 > t_3$, and there is no positive solution; see [12, Lemma 4].

4. Existence Result

Let \mathcal{B} denote the Banach space $C[\rho(t_1), t_n]$ with the norm $\|y\| = \sup_{t \in [\rho(t_1), t_n]} |y(t)|$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{y \in \mathcal{B} : y(t) \geq \Gamma \|y\| \text{ on } [\rho(t_1), t_n]\},$$

where Γ is given in (3.9). The following is a generalization of the discussion found in [19] to arbitrary time scales. Consider the related boundary value problem

$$\begin{aligned} (py^{\nabla})^{\Delta}(t) - q(t)y(t) + f_w(t, y(t)) &= 0, \quad t_1 < t < t_n, \\ \alpha y(t_1) - \beta p(t_1)y^{\nabla}(t_1) &= \sum_{i=2}^{n-1} a_i y(t_i), \quad \gamma y(t_n) + \delta p(t_n)y^{\nabla}(t_n) = \sum_{i=2}^{n-1} b_i y(t_i), \end{aligned}$$

where

$$f_w(t, y(t)) := f(t, y_w(t)) + u(t), \quad y_w(t) := \max\{y(t) - w(t), 0\} \quad (4.1)$$

such that w given in (3.11) is the solution of (3.1), (1.2).

For any fixed $y \in \mathcal{P}$, $y_w \leq y \leq \|y\|$ and by (1.5),

$$\begin{aligned} \int_{t_1}^{t_n} G(t, s) f_w(s, y(s)) \Delta s &\leq \int_{t_1}^{t_n} G(s, s) (z(s) h(y_w(s)) + u(s)) \Delta s \\ &\leq \left(\max_{0 \leq \tau \leq \|y\|} h(\tau) + 1 \right) \int_{t_1}^{t_n} G(s, s) (z(s) + u(s)) \Delta s < \infty. \end{aligned}$$

Additionally, using the properties of Green's function (3.8), for $i = 2, \dots, n - 1$ we have

$$\int_{t_1}^{t_n} G(t_i, s)(z(s) + u(s))\Delta s \leq \int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s.$$

Thus it follows that, for A and B as in (3.12) and (3.13), respectively,

$$\begin{aligned} A(z + u) &= \frac{1}{D} \left| \begin{array}{cc} \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i, s)(z(s) + u(s))\Delta s & d - \sum_{i=2}^{n-1} a_i \phi(t_i) \\ \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i, s)(z(s) + u(s))\Delta s & - \sum_{i=2}^{n-1} b_i \phi(t_i) \end{array} \right| \\ &\leq \frac{1}{D} \left| \begin{array}{cc} \sum_{i=2}^{n-1} a_i & d - \sum_{i=2}^{n-1} a_i \phi(t_i) \\ \sum_{i=2}^{n-1} b_i & - \sum_{i=2}^{n-1} b_i \phi(t_i) \end{array} \right| \int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s \\ &= A \int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s < \infty \end{aligned}$$

and

$$\begin{aligned} B(z + u) &= \frac{1}{D} \left| \begin{array}{cc} - \sum_{i=2}^{n-1} a_i \psi(t_i) & \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i, s)(z(s) + u(s))\Delta s \\ d - \sum_{i=2}^{n-1} b_i \psi(t_i) & \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i, s)(z(s) + u(s))\Delta s \end{array} \right| \\ &\leq \frac{1}{D} \left| \begin{array}{cc} - \sum_{i=2}^{n-1} a_i \psi(t_i) & \sum_{i=2}^{n-1} a_i \\ d - \sum_{i=2}^{n-1} b_i \psi(t_i) & \sum_{i=2}^{n-1} b_i \end{array} \right| \int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s \\ &= B \int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s < \infty. \end{aligned}$$

In the rest of the discussion we make the additional assumption that

$$\int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s < \infty. \tag{4.2}$$

This allows us to define for $y \in \mathcal{P}$ the operator $T : \mathcal{P} \rightarrow \mathcal{B}$ by

$$(Ty)(t) := \lambda \left(\int_{t_1}^{t_n} G(t, s) f_w(s, y(s)) \Delta s + A(f_w) \psi(t) + B(f_w) \phi(t) \right) \quad (4.3)$$

using (3.12), (3.13), and (4.1).

Lemma 4.1. Assume that (1.3), (1.4), (1.5), (3.14), and (4.2) hold. Then $T : \mathcal{P} \rightarrow \mathcal{P}$ is completely continuous.

Proof. For any $y \in \mathcal{P}$, (3.8) and Lemma 3.2 imply that

$$(Ty)(t) \leq \lambda \left(\int_{t_1}^{t_n} G(s, s) f_w(s, y(s)) \Delta s + A(f_w) \psi(t_n) + B(f_w) \phi(\rho(t_1)) \right).$$

On the other hand, from Lemma 3.2 and Lemma 3.3,

$$\begin{aligned} (Ty)(t) &\geq \Gamma \lambda \int_{t_1}^{t_n} G(s, s) f_w(s, y(s)) \Delta s + \lambda (\beta A(f_w) + \delta B(f_w)) \\ &\geq \Gamma \lambda \left(\int_{t_1}^{t_n} G(s, s) f_w(s, y(s)) \Delta s + A(f_w) \psi(t_n) + B(f_w) \phi(\rho(t_1)) \right). \end{aligned}$$

Therefore $(Ty)(t) \geq \Gamma \|Ty\|$ on $[\rho(t_1), t_n]$, so that $T(\mathcal{P}) \subseteq \mathcal{P}$. By a standard application of the Arzela–Ascoli theorem, T is completely continuous. \blacksquare

To establish an existence result we will employ the following fixed point theorem due to Krasnoselskii [10], and seek a fixed point of T in \mathcal{P} .

Theorem 4.2. Let E be a Banach space, $P \subseteq E$ be a cone, and suppose that $\mathcal{S}_1, \mathcal{S}_2$ are bounded open balls of E centered at the origin with $\overline{\mathcal{S}}_1 \subset \mathcal{S}_2$. Suppose further that $L : P \cap (\overline{\mathcal{S}}_2 \setminus \mathcal{S}_1) \rightarrow P$ is a completely continuous operator such that either

- (i) $\|Ly\| \leq \|y\|$, $y \in P \cap \partial\mathcal{S}_1$ and $\|Ly\| \geq \|y\|$, $y \in P \cap \partial\mathcal{S}_2$, or
- (ii) $\|Ly\| \geq \|y\|$, $y \in P \cap \partial\mathcal{S}_1$ and $\|Ly\| \leq \|y\|$, $y \in P \cap \partial\mathcal{S}_2$

holds. Then L has a fixed point in $P \cap (\overline{\mathcal{S}}_2 \setminus \mathcal{S}_1)$.

Theorem 4.3. Assume that (1.3), (1.4), (1.5), (3.14), and (4.2) hold. Then there exists $\lambda^* > 0$ such that the second-order n -point time scale boundary value problem (1.1), (1.2) has at least one positive solution in \mathcal{P} for any $\lambda \in (0, \lambda^*)$.

Proof. By Lemma 4.1, $T : \mathcal{P} \rightarrow \mathcal{P}$ given by (4.3) is completely continuous. Take $\mathcal{S}_1 := \{y \in \mathcal{B} : \|y\| < \xi\}$, for ξ given (3.15). For Γ as in (3.9), let

$$\lambda^* := \min \left\{ 1, \frac{\int_{t_1}^{t_n} G(s, s) u(s) \Delta s}{\Gamma \left(\max_{0 \leq \tau \leq \xi} h(\tau) + 1 \right) \int_{t_1}^{t_n} G(s, s) (z(s) + u(s)) \Delta s} \right\}.$$

Then for any $y \in \mathcal{P} \cap \partial\mathcal{S}_1$,

$$0 \leq y_w(s) \leq y(s) \leq \|y\| = \xi, \quad s \in [\rho(t_1), t_n],$$

and, for A and B as in the statement of Lemma 3.5,

$$\begin{aligned} (Ty)(t) &\leq \lambda \int_{t_1}^{t_n} G(s, s)(z(s)h(y_w(s)) + u(s))\Delta s \\ &\quad + \lambda (A\psi(t_n) + B\phi(\rho(t_1))) \int_{t_1}^{t_n} G(s, s)(z(s)h(y_w(s)) + u(s))\Delta s \\ &\leq \lambda (1 + A\psi(t_n) + B\phi(\rho(t_1))) \left(\max_{0 \leq \tau \leq \xi} h(\tau) + 1 \right) \\ &\quad \times \int_{t_1}^{t_n} G(s, s)(z(s) + u(s))\Delta s \\ &\leq \xi = \|y\|. \end{aligned}$$

Hence $\|Ty\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial\mathcal{S}_1$. Pick $K \in \mathbb{R}$ such that $K > 0$ and

$$1 \leq \frac{\lambda K \Gamma}{\xi + 1} \min_{t_2 \leq t \leq t_3} \int_{t_2}^{t_3} G(t, s)\Delta s.$$

By (1.5), for any $t \in [t_2, t_3]_{\mathbb{T}}$, there exists a constant $N > 0$ such that $f(t, y) > Ky$ for $y > N$. Pick $Q := \max \left\{ \lambda(\xi + 1), \xi + 1, \frac{N(\xi + 1)}{\Gamma} \right\}$. If $\mathcal{S}_2 := \{y \in \mathcal{B} : \|y\| < Q\}$, then for any $y \in \mathcal{P} \cap \partial\mathcal{S}_2$ and $t \in [\rho(t_1), t_n]_{\mathbb{T}}$,

$$\begin{aligned} y(t) - w(t) &\geq y(t) - \lambda \Gamma \xi \geq y(t) - \frac{\lambda \xi}{Q} y(t) \geq \left(1 - \frac{\lambda \xi}{Q} \right) y(t) \\ &\geq \left(1 - \frac{\lambda \xi}{\lambda(\xi + 1)} \right) y(t) = \frac{y(t)}{\xi + 1} \geq 0. \end{aligned}$$

Thus

$$\min_{t \in [t_2, t_3]_{\mathbb{T}}} (y(t) - w(t)) \geq \min_{t \in [t_2, t_3]_{\mathbb{T}}} \frac{y(t)}{\xi + 1} \geq \frac{\Gamma Q}{\xi + 1} \geq N,$$

so that

$$\begin{aligned}
\min_{t \in [t_2, t_3]_{\mathbb{T}}} (Ty)(t) &= \min_{t \in [t_2, t_3]_{\mathbb{T}}} \lambda \left(\int_{t_1}^{t_n} G(t, s) f_w(s, y(s)) \Delta s + A(f_w) \psi(t) + B(f_w) \phi(t) \right) \\
&\geq \lambda \min_{t \in [t_2, t_3]_{\mathbb{T}}} \int_{t_1}^{t_n} G(t, s) f_w(s, y(s)) \Delta s \\
&\geq \lambda \min_{t \in [t_2, t_3]_{\mathbb{T}}} \int_{t_2}^{t_3} G(t, s) f_w(s, y(s)) \Delta s \\
&\geq \lambda K \min_{t \in [t_2, t_3]_{\mathbb{T}}} \int_{t_2}^{t_3} G(t, s) (y(s) - w(s)) \Delta s \\
&\geq \frac{\lambda K \Gamma Q}{\xi + 1} \min_{t \in [t_2, t_3]_{\mathbb{T}}} \int_{t_2}^{t_3} G(t, s) \Delta s \\
&= \frac{\lambda K \Gamma \|y\|}{\xi + 1} \min_{t \in [t_2, t_3]_{\mathbb{T}}} \int_{t_2}^{t_3} G(t, s) \Delta s \geq \|y\|.
\end{aligned}$$

Hence for $y \in \mathcal{P} \cap \partial \mathcal{S}_2$ we have $\|Ty\| \geq \|y\|$. By Theorem 4.2, T has a fixed point y such that $\xi \leq \|y\| \leq Q$. But then

$$y(t) - w(t) \geq \Gamma \xi - \lambda \Gamma \xi \geq (1 - \lambda) \Gamma \xi \geq 0.$$

As a consequence, this y solves the boundary value problem

$$\begin{aligned}
(py^\nabla)^\Delta(t) - q(t)y(t) + \lambda(f(t, y(t) - w(t)) + u(t)) &= 0, \quad t \in (t_1, t_n)_{\mathbb{T}}, \\
\alpha y(t_1) - \beta p(t_1)y^\nabla(t_1) &= \sum_{i=2}^{n-1} a_i y(t_i), \quad \gamma y(t_n) + \delta p(t_n)y^\nabla(t_n) = \sum_{i=2}^{n-1} b_i y(t_i).
\end{aligned}$$

Now set $x(t) := y(t) - w(t)$ for w given in (3.11). Then $y^\nabla = x^\nabla + w^\nabla$ and $(py^\nabla)^\Delta = (px^\nabla)^\Delta + (pw^\nabla)^\Delta$. As w is the solution of (3.1), (1.2), we see that

$$\begin{aligned}
(px^\nabla)^\Delta(t) - q(t)x(t) + \lambda f(t, x(t)) &= 0, \quad t \in (t_1, t_n)_{\mathbb{T}}, \\
\alpha x(t_1) - \beta p(t_1)x^\nabla(t_1) &= \sum_{i=2}^{n-1} a_i x(t_i), \quad \gamma x(t_n) + \delta p(t_n)x^\nabla(t_n) = \sum_{i=2}^{n-1} b_i x(t_i),
\end{aligned}$$

in other words, x is a positive solution of the second-order n -point time scale boundary value problem (1.1), (1.2). ■

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