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An Extension of the Fixed Point Theorem of Cone Expansion and Compression of Functional Type

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Abstract

The fixed point theorem of cone expansion and compression of norm type is extended with a comparison to monotonic operators on intervals which yields improved existence results when applicable. We conclude with an application verifying the existence of a positive solution to a second order right focal boundary value problem.

Key words: fixed point theorems, boundary value problem, differential equations.
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1 Preliminaries

The fixed point theorem of cone expansion and compression of functional type [3] is an extension of the fixed point theorem of cone expansion and compression of norm type that is usually referred to as Krasnoselskii's fixed point theorem, a proof of which can be found in [5]. In this paper we will extend the fixed point theorem of cone expansion and compression of functional type by comparing to monotonic operators on an appropriate order interval using techniques similar to those by Avery and Henderson used in the extension of the five functionals fixed point theorem [4]. The bounds of the nonlinear term on certain intervals in conjuction with standard existence techniques result in the construction of the order interval as well as the comparing operators. In the application section we demonstrate the technique for a second order right focal boundary value problem and conclude with an example. Existence of one and two positive solutions for right focal boundary value problems (second order and higher) has been studied extensively. A representative of papers in this area include Henderson [11], Eloe and Henderson [7], Erbe [8], Erbe and Wang [10], Erbe, Hu and Wang [9], Agarwal and Wong [1], and Merdivenci [12, 13]. In this section we will state the definitions that are used in the remainder of the paper.

Definition 1.1 *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:*

(i) $x \in P, \lambda \geq 0$ *implies* $\lambda x \in P$*;*

(ii) $x \in P, -x \in P$ *implies* $x = 0$*.*

Every cone $P \subset E$ induces an ordering, $\leq$, in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 1.2 *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Definition 1.3 *Let P be a cone in a real Banach space E and $D \subseteq E$. Then the operator $A : D \to E$ is said to be increasing on D provided $x_1, x_2 \in D$ with $x_1 \leq x_2$ implies $Ax_1 \leq Ax_2$.*

The existence of increasing completely continuous operators on portions of the cone that satisfy certain properties is the basis of the extension of the Fixed Point Theorem of Cone Expansion and Compression of Functional Type.

Definition 1.4 *A cone P of a real Banach space E is said to be normal if there exists a positive constant δ such that $\|x+y\| \geq \delta$ for all $x, y \in P$ with $\|x\| = \|y\| = 1$.*

The following theorem is an elementary fact about normal cones. A proof can be found in [6].

Theorem 1.1 *Let P be a normal cone in a real Banach space E. The cone P is normal if and only if the norm of the Banach space E is semi-monotone; that is, there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies that $\|x\| \leq N\|y\|$.*

The following theorems concerning the convergence of Picard iterates are restatements of theorems that can be found in [6].

Theorem 1.2 *Let P be a normal cone in a real Banach space E and $A : P \to E$ be an increasing, completely continuous operator. If $u \in P$ with $Au \leq u$ and there exists a $v \in P$ such that $v \leq A^n u$ for all $n \in N$ then*

$$\{A^n u\}_{n=1}^{\infty}$$

is a decreasing sequence bounded below by $v \in P$ and there exists a $u^* \in P$ such that

$$u^* = \lim_{n\to\infty} A^n u$$

with

$$v \leq u^* \leq A^n u \leq A^{n-1} u \leq \cdots \leq Au.$$

Theorem 1.3 *Let P be a normal cone in a real Banach space E and $A : P \to E$ be an increasing, completely continuous operator. If $u \in P$ with $u \leq Au$ and there exists a $v \in P$ such that $v \geq A^n u$ for all $n \in N$ then*

$$\{A^n u\}_{n=1}^{\infty}$$

is an increasing sequence bounded above by $v \in P$ and there exists a $u^* \in P$ such that

$$u^* = \lim_{n\to\infty} A^n u$$

with

$$v \geq u^* \geq A^n u \geq A^{n-1} u \geq \cdots \geq Au.$$

Definition 1.5 *A map α is said to be a* nonnegative continuous functional *on a cone P of a real Banach space E if*

$$\alpha : P \to [0, \infty)$$

is continuous.

Let α and γ be nonnegative continuous functionals on P; then, for positive real numbers r and R, we define the following sets:

$$P(\gamma, R) = \{x \in P : \gamma(x) < R\} \tag{1}$$

and

$$P(\gamma, \alpha, r, R) = \{x \in P : r < \alpha(x) \text{ and } \gamma(x) < R\}. \tag{2}$$

2 Fixed Point Theorems

The following theorem is the extension of the Fixed Point Theorem of Cone Expansion and Compression of Functional Type.

Theorem 2.1 *Let P be a normal cone in a real Banach space E, and let α and γ be nonnegative continuous functionals on P. Assume $P(\gamma, \alpha, r, R)$ as in (2) is a nonempty bounded subset of P,*

$$A : \overline{P(\gamma, \alpha, r, R)} \to P$$

is a completely continuous operator and

$$\overline{P(\alpha, r)} \subseteq P(\gamma, R)$$

for these sets as in (1). If one of the two conditions

$(H1)$ *$\alpha(Ax) \le r$ for all $x \in \partial P(\alpha, r)$, $\gamma(Ax) \ge R$ for all $x \in \partial P(\gamma, R)$,*

$$\inf_{x \in \partial P_X(\gamma, R)} \|Ax\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \ge 1$, and $\mu \in (0, 1]$ the functionals satisfy the properties

$$\alpha(\lambda y) \ge \lambda \alpha(y), \;\; \gamma(\mu z) \le \mu \gamma(z), \text{ and } \alpha(0) = 0;$$

or

$(H2)$ *$\alpha(Ax) \ge r$ for all $x \in \partial P(\alpha, r)$, $\gamma(Ax) \le R$ for all $x \in \partial P(\gamma, R)$,*

$$\inf_{x \in \partial P_X(\alpha, r)} \|Ax\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$ and $\mu \ge 1$ the functionals satisfy the properties

$$\alpha(\lambda y) \le \lambda \alpha(y), \;\; \gamma(\mu z) \ge \mu \gamma(z), \text{ and } \gamma(0) = 0$$

is satisfied, then A has at least one positive fixed point x^* such that

$$r \leq \alpha(x^*) \ \ \text{and} \ \ \gamma(x^*) \leq R.$$

Moreover, suppose there exist $x_l, x_u \in P$ such that $\overline{P(\gamma, \alpha, r, R)} \subseteq [x_l, x_u]$.

$(E1)$ *If there exists an increasing completely continuous operator $U : [x_l, x_u] \to P$ such that $Ax \leq Ux$ for all $x \in [x_l, x_u]$ and $U^2 x_u \leq U x_u$, then*

$$x^* \leq x_u^* \leq U^n x_u,$$

where $n \in \mathbb{N}$ and $x_u^* = \lim_{n\to\infty} U^n x_u$.

$(E2)$ *If there exists an increasing completely continuous operator $L : [x_l, x_u] \to P$ such that $Lx \leq Ax$ for all $x \in [x_l, x_u]$ and $Lx_l \leq L^2 x_l$, then*

$$L^n x_l \leq \ x_l^* \leq x^*,$$

where $n \in \mathbb{N}$ and $x_l^* = \lim_{n\to\infty} L^n x_l$.

Proof: The fixed point theorem of cone expansion and compression of functional type [3] guarantees the existence of a fixed point $x^* \in P(\gamma, \alpha, r, R)$ for the operator A. We are left to prove the extensions, statements $(E1)$ and $(E2)$. We will prove $(E1)$ but omit the proof of $(E2)$, which is nearly identical. Suppose there exist $x_l, x_u \in P$ such that $\overline{P(\gamma, \alpha, r, R)} \subseteq [x_l, x_u]$ and an increasing completely continuous operator

$$U : [x_l, x_u] \to P$$

such that

$$Ax \leq Ux, \quad U^2 x_u \leq U x_u$$

for all $x \in [x_l, x_u]$. Then

$$x^* = Ax^* \leq Ux^* \leq Ux_u.$$

Since U is an increasing operator it follows by induction that, for all natural numbers n, we have

$$x^* \leq U^n x_u \leq U^{n-1} x_u \leq \cdots \leq U x_u.$$

Thus

$$\{U^n x_u\}_{n=1}^{\infty}$$

is a decreasing sequence in P bounded below by x^*. By Theorem 1.2 there exists an x_u^* such that

$$x_u^* = \lim_{n\to\infty} U^n x_u \quad \text{with} \quad x^* \leq x_u^* \leq U^n x_u$$

for all natural numbers n. □

3 Application

Consider the second-order nonlinear focal boundary-value problem

$$y''(t) + f(y(t)) = 0, \quad t \in (0,1), \tag{3}$$

$$y(0) = 0 = y'(1), \tag{4}$$

where $f : \mathbb{R} \to [0,\infty)$ is continuous. The solutions of (3), (4) are the fixed points of the operator A defined on the Banach space $C[0,1]$ (with our norm $\|\cdot\|$ being the sup norm) by

$$Ay(t) := \int_0^1 G(t,s) f(y(s))ds, \quad t \in [0,1],$$

where

$$G(t,s) = \begin{cases} t & : \ t \le s \\ s & : \ s \le t \end{cases}$$

is Green's function for the operator L defined by

$$Ly(t) := -y''$$

with right-focal boundary conditions

$$y(0) = 0 = y'(1).$$

Throughout this section of the paper we will use the facts that $G(t,s)$ is nonnegative and for each fixed $s \in [0,1]$ Green's function is nondecreasing in t.

Define the cone $P \subset E = C[0,1]$ by

$$P := \{y \in E \ : \ \text{y is concave, nonnegative, and non-decreasing}\};$$

then clearly $A : P \to P$ by the properties of Green's function, and by the concavity for all $y \in P$ we have

$$y(t) \ge t\|y\|.$$

Let $\eta \in (0,1)$ and define the functionals $\alpha, \gamma : P \to \mathbb{R}$ by

$$\alpha(y) := \min_{t \in [\eta,1]} y(t) = y(\eta)$$

and

$$\gamma(y) := \max_{t \in [0,1]} y(t) = \|y\| = y(1).$$

In the following theorem, using the standard technique of bounding the nonlinearity by constants we show how to employ the extension.

Theorem 3.1 *Suppose there exist positive real numbers r and R with $r \leq \eta R$, and continuous $f : \mathbb{R} \to [0, \infty)$, such that the following conditions are met:*

(i) $f(w) \leq 2R$ *for* $w \in [0, R]$,

(ii) $f(w) \geq \frac{r}{\eta(1-\eta)}$ *for* $w \in [r, R]$.

Then, the boundary value problem (3), (4) *has at least one positive solution* x^* *such that*

$$r \leq \alpha(x^*) = x^*(\eta) \leq x^*(1) = \gamma(x^*) \leq R.$$

Moreover,

$$rt/\eta \leq x^*(t) \leq Rt(2-t), \quad t \in [0, \eta]$$

and

$$\frac{r}{\eta(1-\eta)} \left(t - \frac{1}{2}(t^2 + \eta^2) \right) \leq x^*(t) \leq Rt(2-t), \quad t \in [\eta, 1].$$

Proof: A simple application of Theorem 1.1 verifies that the cone P is normal. If $y \in \overline{P(\alpha, r)}$, then $r \geq y(\eta) \geq \eta y(1) = \eta\gamma(y)$. It follows that $R \geq r/\eta \geq \gamma(y)$, thus $y \in P(\gamma, R)$ and we have that $\overline{P(\alpha, r)} \subseteq P(\gamma, R)$. Clearly, for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$ and $\mu \geq 1$ the functionals satisfy the properties

$$\alpha(\lambda y) \leq \lambda\alpha(y), \;\; \gamma(\mu z) \geq \mu\gamma(z), \text{ and } \; \gamma(0) = 0.$$

Claim 1. $\alpha(Ay) \geq r$ for all $y \in \partial P(\alpha, r)$:
If $y \in \partial P(\alpha, r)$, then

$$\begin{aligned} \alpha(Ay) &= Ay(\eta) = \int_0^1 G(\eta, s) f(y(s)) ds \geq \int_\eta^1 G(\eta, s) f(y(s)) ds \\ &\geq \frac{r}{\eta(1-\eta)} \int_\eta^1 G(\eta, s)\, ds = r. \end{aligned}$$

This also verifies that

$$\inf_{y \in \partial P(\alpha, r)} \|Ay\| \geq r > 0.$$

Claim 2. $\gamma(Ay) \leq R$ for all $y \in \partial P(\gamma, R)$:
If $y \in \partial P(\gamma, R)$, then

$$\gamma(Ay) = Ay(1) = \int_0^1 G(1, s) f(y(s)) ds \leq 2R \int_0^1 G(1, s) ds = R.$$

Therefore the hypotheses of the fixed point theorem of cone expansion and compression of functional type are satisfied and hence the boundary value problem (3), (4) has at least one positive solution x^* such that

$$r \leq \alpha(x^*) = x^*(\eta) \leq x^*(1) = \gamma(x^*) \leq R.$$

If we define the increasing (since they are constant) operators $L : P \to P$ and $U : P \to P$ by

$$Ly(t) := \int_\eta^1 G(t,s)\frac{r}{\eta(1-\eta)}ds \quad \text{and} \quad Uy(t) := \int_0^1 G(t,s)2Rds,$$

then for all $y \in P(\gamma, \alpha, r, R)$ we have

$$Ly \le Ay \le Uy.$$

Moreover, if we define $y_u, y_l \in P$ by $y_u(s) := R$ and

$$y_l(s) := \begin{cases} \frac{rs}{\eta} & : s \le \eta \\ r & : \eta \le s, \end{cases}$$

then as a consequence of the concavity of P we have $P(\gamma, \alpha, r, R) \subseteq [y_l, y_u]$. By the same argument as in Claim 2 we have that $Uy_u \le y_u$ and using the concavity of Ly_l together with $Ly_l(\eta) \ge r$ and $Ly_l(1) \ge r$ we have $Ly_1 \ge y_l$. Hence, by the extension of the fixed point theorem of cone expansion and compression of functional type we have

$$Ly_l(t) = \frac{r}{\eta(1-\eta)}\int_\eta^1 G(t,s)ds \le x^*(t) \le 2R\int_0^1 G(t,s)ds = Uy_u(t).$$

Simple calculations show that $\int_0^1 G(t,s)ds = t - \frac{1}{2}t^2$, and

$$\int_\eta^1 G(t,s)ds = \begin{cases} t(1-\eta) & : \ t \in [0,\eta] \\ t - \frac{1}{2}(t^2+\eta^2) & : \ t \in [\eta, 1]. \end{cases}$$

□

In the following theorem, given more information concerning the nonlinearity, we show how to take advantage of the Picard iterates.

Theorem 3.2 *Suppose there exist positive real numbers r and R with $r \le \eta R$, and continuous $f : \mathbb{R} \to [0, \infty)$, such that*

$F(i)$ $my + b \le f(y)$ *for* $y \in [r, R]$,

$F(ii)$ $f(y) \le My + B$ *for* $y \in [0, R]$,

and the non-negative constants $m, b, M,$ and B satisfy the following conditions:

$C(i)$ $r \le \eta(1-\eta)(mr + b)$,

$C(ii)$ $B \leq (2-M)R$

$C(iii)$ $R(m-M) \leq (B-b)$.

Then, the boundary value problem (3), (4) *has at least one positive solution* y^* *such that*

$$r \leq \alpha(y^*) = y^*(\eta) \leq y^*(1) = \gamma(y^*) \leq R.$$

Moreover, for all natural numbers n *we have*

$$L^n y_l \leq y^*(t) \leq U^n y_u,$$

where the functions y_l *and* y_u *are defined in* (5) *and the operators* L *and* U *are defined in* (6).

Proof: A simple application of Theorem 1.1 verifies that the cone P is normal. If $y \in \overline{P(\alpha, r)}$, then $r \geq y(\eta) \geq \eta y(1) = \eta\gamma(y)$. Consequently $R \geq \frac{r}{\eta} \geq \gamma(y)$, so that $y \in P(\gamma, R)$ and we have that $\overline{P(\alpha, r)} \subseteq P(\gamma, R)$. Clearly, for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$ and $\mu \geq 1$ the functionals satisfy the properties

$$\alpha(\lambda y) \leq \lambda\alpha(y), \;\; \gamma(\mu z) \geq \mu\gamma(z), \text{ and } \; \gamma(0) = 0.$$

Define $y_u, y_l \in P$ for $s \in [0, 1]$ by

$$y_u(s) := R, \qquad y_l(s) := \begin{cases} \frac{rs}{\eta} & : s \leq \eta \\ r & : \eta \leq s. \end{cases} \tag{5}$$

Claim 1. $\alpha(Ay) \geq r$ for all $y \in \partial P(\alpha, r)$:
If $y \in \partial P(\alpha, r)$, then by the concavity of P we have $y \geq y_l$ and

$$\begin{aligned} \alpha(Ay) &= Ay(\eta) = \int_0^1 G(\eta, s) f(y(s)) ds \overset{F(i)}{\geq} \int_\eta^1 G(\eta, s)(my(s) + b) ds \\ &\geq \int_\eta^1 \eta(mr + b) ds = \eta(1-\eta)(mr + b) \overset{C(i)}{\geq} r. \end{aligned}$$

This also verifies that

$$\inf_{y \in \partial P(\alpha, r)} \|Ay\| \geq r > 0.$$

Claim 2. $\gamma(Ay) \leq R$ for all $y \in \partial P(\gamma, R)$:
If $y \in \partial P(\gamma, R)$, then

$$\begin{aligned} \gamma(Ay) &= Ay(1) = \int_0^1 G(1, s) f(y(s)) ds \overset{F(ii)}{\leq} \int_0^1 G(1, s)(My(s) + B) ds \\ &\leq \int_0^1 G(1, s)(MR + B) ds = \frac{MR + B}{2} \overset{C(ii)}{\leq} R. \end{aligned}$$

Therefore the hypotheses of the fixed point theorem of cone expansion and compression of functional type are satisfied and hence the boundary value problem (3), (4) has at least one positive solution x^* such that

$$r \leq \alpha(y^*) = y^*(\eta) \leq y^*(1) = \gamma(y^*) \leq R.$$

If we define the increasing operators $L : P \to P$ and $U : P \to P$ by

$$Ly(t) := \int_\eta^1 G(t,s)(my(s)+b)ds \quad \text{and} \quad Uy(t) := \int_0^1 G(t,s)(My(s)+B)ds, \tag{6}$$

then for all $y \in P(\gamma, \alpha, r, R)$ we have

$$Ly \leq Ay \leq Uy.$$

As a consequence of the concavity of P we have $P(\gamma, \alpha, r, R) \subseteq [y_l, y_u]$. By the same argument as in Claim 2 we have that $Uy_u \leq y_u$, and using the concavity of Ly_l together with $Ly_l(\eta) \geq r$ and $Ly_l(1) \geq r$ we have $Ly_1 \geq y_l$. Hence, by the extension of the fixed point theorem of cone expansion and compression of functional type we have for all natural numbers n that

$$L^n y_l(t) \leq y^*(t) \leq U^n y_u(t).$$

□

Example: The function

$$f(y) = \frac{y^2}{2} - \frac{y^3}{24} + 5$$

satisfies the hypotheses of the previous theorem with $r = 1$, $R = 10$, and $\eta = 1/2$ using the straight-line comparisons

$$\begin{cases} y + 3 \leq f(y) & : \ y \in [1, 10], \\ f(y) \leq y + 10 & : \ y \in [0, 10]. \end{cases}$$

Therefore, the boundary value problem

$$y''(t) + \frac{y(t)^2}{2} - \frac{y(t)^3}{24} + 5 = 0, \quad t \in (0,1), \tag{7}$$

$$y(0) = 0 = y'(1) \tag{8}$$

has a positive solution y^* such that:

$$L^n y_l(t) \leq y^*(t) \leq U^n y_u(t), \quad t \in [0,1],$$

where

$$Ly_l(t) = \begin{cases} 2t & : \ 0 \le t \le 1/2 \\ 4t - 2t^2 - 1/2 & : \ 1/2 \le t \le 1 \end{cases} \quad \text{and} \quad Uy_u(t) = 10t(2-t).$$

For example, at $t = 1$, we have the following table using *Mathematica* 5.1:

n	$L^n y_l(1)$	$U^n y_u(1)$
1	1.5	10
2	1.6354	9.1666
3	1.6803	8.7777
4	1.6954	8.6175
5	1.7005	8.5525
6	1.7022	8.5261
7	1.7028	8.5154
8	1.7030	8.5111
9	1.7031	8.5093
10	1.7031	8.5086

Note that an application of the fixed point theorem of cone expansion and compression of functional type to this problem would have only yielded the existence of a solution greater than one on the right half of the interval and less than ten on the entire interval. Other results in the literature applying fixed point theorems proven using degree theory have similar conclusions.

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