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Eigenvalue Intervals for Even-Order Sturm-Liouville Dynamic Equations

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Abstract

We study the existence of eigenvalue intervals for the even-order dynamic equation on time scales,

$$(-1)^n x^{(\Delta\nabla)^n}(t) = \lambda c(t) f(x(t)), \quad t \in [a, b]$$

satisfying the boundary conditions

$$\alpha_{i+1} x^{(\Delta\nabla)^i}(a) - \beta_{i+1} x^{(\Delta\nabla)^i\Delta}(a) = 0, \quad \gamma_{i+1} x^{(\Delta\nabla)^i}(b) + \delta_{i+1} x^{(\Delta\nabla)^i\Delta}(b) = 0$$

for $0 \le i \le n-1$, where f is a positive function and c is a nonnegative function that is allowed to vanish on some subintervals of $[a, b]$ of the time scale. The methods involve applications of Krasnoselskii's fixed-point theorem for operators on a cone in a Banach space.

Key words: Green's function, boundary value problem, time scales, cone.
AMS Subject Classification: 34B15, 34B10, 34B18.

1 Introduction

In this paper we are concerned with the existence of eigenvalues for the even-order dynamic equation on time scales

$$(-1)^n x^{(\Delta\nabla)^n}(t) = \lambda c(t) f(x(t)), \quad t \in [a, b] \tag{1}$$

satisfying Sturm-Liouville-like boundary conditions

$$\alpha_{i+1} x^{(\Delta\nabla)^i}(a) - \beta_{i+1} x^{(\Delta\nabla)^i\Delta}(a) = 0, \quad \gamma_{i+1} x^{(\Delta\nabla)^i}(b) + \delta_{i+1} x^{(\Delta\nabla)^i\Delta}(b) = 0. \tag{2}$$

Here $n \geq 1$ and $0 \leq i \leq n-1$, with $a \in \mathbb{T}_{\kappa^n}$, $b \in \mathbb{T}^{\kappa^n}$ for a time scale $\mathbb{T}$, and $\sigma^n(a) < \rho^n(b)$. We take $\alpha_j, \beta_j, \gamma_j, \delta_j \geq 0$ and

$$d_j := \gamma_j \beta_j + \alpha_j \delta_j + \alpha_j \gamma_j (b-a) > 0.$$

A solution $x \in C_{ld}^{(2n)}[a,b]$ of (1), (2) is defined on $[\rho^n(a), \sigma^n(b)]$; the solution is positive if it satisfies (1), (2), is nonnegative and is not identically zero on $[a,b]$.

There has been much interest recently in this area of obtaining optimal eigenvalue intervals of boundary value problems, often using Krasnoselskii [25] fixed point theorems to obtain intervals based on positive solutions inside a cone. A few papers along these lines are Agarwal, Bohner, and Wong [3], Anderson and Davis [5], Davis, Henderson, Prasad, and Yin [14], Eloe and Henderson [15], Erbe, Hu, and Wang [20], Erbe and Tang [21], Henderson and Wang [23], Jiang and Liu [24], Wong and Agarwal [26].

When seeking eigenintervals of boundary value problems for dynamic equations on time scales, many of these same methods carry over; see Agarwal, Bohner, and Wong [2], Anderson [4, 6], Chyan, Davis, Henderson, and Yin [11], Chyan and Henderson [12], Chyan, Henderson, and Pan [13], and Erbe and Peterson [17, 18, 19], for example.

The next two sections introduce the basic notation and concepts for time scales, and the Green's function for the boundary value problem in question.

2 Preliminaries About Time Scales

A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. Hilger [22] initially introduced time scales with the twin goals of unifying the continuous and discrete calculus and extending the results to a dynamic calculus for general time scales. Some other early papers in this area include Agarwal and Bohner [1], Atici and Guseinov [7], Aulbach and Hilger [8], and Erbe and Hilger [16]. For an excellent introduction to the overall area of dynamic equations on time scales, see the recent texts by Bohner and Peterson [9, 10], from which we cull the following definitions. The functions $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are jump operators given by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$). The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [9] of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, we have $f^\Delta = f'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have the forward difference operator, $f^\Delta(t) = f(t+1) - f(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [7, 10] of f at t, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, we have $f^\nabla = f'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have the backward difference operator, $f^\nabla(t) = f(t) - f(t-1)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous on $[a, b]$, denoted $f \in C_{ld}[a, b]$, provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous. It is known [7, 10] that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

If $\mathbb{T} = \mathbb{R}$, for example, then

$$\int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

with the right-hand side representing the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$ for some $h > 0$, then

$$\int_a^b f(t)\nabla t = \begin{cases} \sum_{k=a/h+1}^{b/h} hf(kh) & : a < b \\ 0 & : a = b \\ -\sum_{k=b/h+1}^{a/h} hf(kh) & : b < a. \end{cases}$$

Throughout this paper, we use the time scale interval

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

3 Green's Function

Shortly we will be concerned with a completely continuous operator whose kernel is Green's function for the related homogeneous problem

$$(-1)^n x^{(\Delta\nabla)^n}(t) = 0, \quad t \in [a,b] \tag{3}$$

satisfying boundary conditions (2). For $1 \le j \le n$, let $G_j(t,s)$ be Green's function for the boundary value problems

$$-x^{\Delta\nabla}(t) = 0, \quad t \in [a,b]$$

$$\alpha_j x(a) - \beta_j x^{\Delta}(a) = 0, \quad \gamma_j x(b) + \delta_j x^{\Delta}(b) = 0.$$

Then, for $1 \le j \le n$,

$$G_j(t,s) = \begin{cases} \frac{1}{d_j}\{\alpha_j(t-a)+\beta_j\}\{\gamma_j(b-s)+\delta_j\} : & t \le s, \\ \frac{1}{d_j}\{\alpha_j(s-a)+\beta_j\}\{\gamma_j(b-t)+\delta_j\} : & s \le t. \end{cases} \tag{4}$$

It is shown in [4] that for all $s,t \in [a,b]$,

$$g_j(t)G_j(s,s) \le G_j(t,s) \le G_j(s,s), \tag{5}$$

where

$$g_j(t) := \min\left\{ \frac{\alpha_j(t-a)+\beta_j}{\alpha_j(b-a)+\beta_j}, \ \frac{\gamma_j(b-t)+\delta_j}{\gamma_j(b-a)+\delta_j} \right\} < 1, \tag{6}$$

for $1 \le j \le n$. Let $H_1(t,s) = G_1(t,s)$, and recursively define

$$H_j(t,s) = \int_a^b H_{j-1}(t,r)G_j(r,s)\nabla r$$

for $2 \le j \le n$. Then $H_n(t,s)$ is Green's function for (3), (2); x is a solution of (1), (2) if and only if

$$x(t) = \lambda \int_a^b H_n(t,s)c(s)f(x(s))\nabla s$$

for a given λ.

Example 3.1 *Let $n = 2$, $\mathbb{T} = \mathbb{Z}$, and consider the discrete, central-difference Lidstone problem*

$$x^{\Delta\nabla\Delta\nabla}(t) = 0, \quad t \in [a,b]$$

$$x(a) = x^{\Delta\nabla}(a) = 0, \quad x(b) = x^{\Delta\nabla}(b) = 0.$$

Because of the Lidstone boundary conditions,

$$H_1(t,s) = G_j(t,s) = G(t,s) := \begin{cases} \frac{(t-a)(b-s)}{b-a} : & t \le s \\ \frac{(s-a)(b-t)}{b-a} : & s \le t \end{cases}$$

and

$$H_2(t,s) = \sum_{r=a+1}^{b} G(t,r)G(r,s) = \begin{cases} u_2(t,s) : & t \le s \\ v_2(t,s) : & s \le t. \end{cases}$$

For $t \le s$*,*

$$\begin{aligned} u_2(t,s) &= \sum_{r=a+1}^{t} \frac{(r-a)(b-t)(r-a)(b-s)}{(b-a)^2} + \sum_{r=t+1}^{s} \frac{(t-a)(b-r)(r-a)(b-s)}{(b-a)^2} \\ &\quad + \sum_{r=s+1}^{b} \frac{(t-a)(b-r)(s-a)(b-r)}{(b-a)^2} \\ &= \frac{(t-a)(b-s)(2bt+1-s^2-t^2-2a(b-s))}{6(b-a)}; \end{aligned}$$

similarly for $t \ge s$*,*

$$v_2(t,s) = \frac{(s-a)(b-t)(2bs+1-t^2-s^2-2a(b-t))}{6(b-a)} = u_2(s,t).$$

Example 3.2 *Let* $n = 2$*,* $\mathbb{T} = [0,1] \cup [2,3]$*, and again consider the Lidstone problem*

$$x^{\Delta\nabla\Delta\nabla}(t) = 0, \quad t \in [0,1] \cup [2,3]$$
$$x(0) = x^{\Delta\nabla}(0) = 0, \quad x(3) = x^{\Delta\nabla}(3) = 0.$$

Because of the Lidstone boundary conditions,

$$H_1(t,s) = G_j(t,s) = G(t,s) := \begin{cases} \frac{t(3-s)}{3} : & t \le s \\ \frac{s(3-t)}{3} : & s \le t \end{cases}$$

and

$$H_2(t,s) = \int_0^3 G(t,r)G(r,s)\nabla r = \begin{cases} u_2(t,s) : & t \le s \\ v_2(t,s) : & s \le t. \end{cases}$$

For $2 < t \le s \le 3$*,*

$$\begin{aligned} u_2(t,s) &= \int_0^t \frac{r^2(3-t)(3-s)}{9}\nabla r + \int_t^s \frac{t(3-r)r(3-s)}{9}dr + \int_s^3 \frac{t(3-r)s(3-r)}{9}dr \\ &= \frac{1}{54}(3-s)[30 - t(10-18s+3s^2) - 3t^3]; \end{aligned}$$

similarly for $3 \geq t \geq s > 2$,

$$v_2(t,s) = \frac{1}{54}(3-t)[30 - s(10 - 18t + 3t^2) - 3s^3].$$

For the rest of the paper we have the assumptions

(A1) c is a nonnegative, left-dense continuous function defined on $[a,b]$ satisfying

$$0 < \int_a^b G_n(s,s)c(s)\nabla s < \infty, \tag{7}$$

where, using (4),

$$G_n(s,s) = \frac{1}{d_n}\left[\alpha_n(s-a) + \beta_n\right]\left[\gamma_n(b-s) + \delta_n\right].$$

(A2) $f : [0,\infty) \to [0,\infty)$ is continuous such that both

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{and} \quad f_\infty := \lim_{x \to \infty} \frac{f(x)}{x}$$

exist.

Let $a < \xi < \omega < b$ be chosen from $\mathbb{T}$ such that

$$\int_\xi^\omega G_n(s,s)c(s)\nabla s > 0; \tag{8}$$

as c is a nonnegative function, this allows c to vanish on some subintervals. Following the ideas of [26] in a related difference equations case, let

$$k_j := \min_{t \in [\xi,\omega]} g_j(t) \tag{9}$$

for g_j as in (6),

$$L_j := \int_a^b G_j(r,r)\nabla r, \quad 1 \leq j \leq n,$$

$$L := \prod_{j=1}^{n-1} L_j,$$

$$M_j := \int_\xi^\omega G_j(r,r)\nabla r, \quad 1 \leq j \leq n,$$

and

$$K := k_n \prod_{j=1}^{n-1} \frac{k_j M_j}{L_j} < 1. \tag{10}$$

Using mathematical induction it is straightforward to see that

$$0 \leq H_n(t,s) \leq LG_n(s,s), \quad t,s \in [a,b] \tag{11}$$

and

$$KLG_n(s,s) \leq H_n(t,s), \quad t \in [\xi,\omega], \quad s \in [a,b]. \tag{12}$$

4 Eigenvalue Intervals

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnoselskii [25].

Theorem 4.1 *Let E be a Banach space, $P \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$

holds. Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $\mathcal{B}$ denote the Banach space $C_{ld}[a,b]$ with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \geq 0 \text{ on } [a,b], \text{ and } x(t) \geq K\|x\| \text{ on } [\xi,\omega]\},$$

where K is given in (10).

Theorem 4.2 *Suppose* $(A1)$ *and* $(A2)$ *hold. Then for each λ satisfying*

$$\frac{1}{f_\infty K^2 L \int_\xi^\omega G_n(s,s)c(s)\nabla s} < \lambda < \frac{1}{f_0 L \int_a^b G_n(s,s)c(s)\nabla s} \tag{13}$$

there exists at least one positive solution of (1), (2) *in $\mathcal{P}$, for ξ, ω as in* (8).

Proof: Let λ be as in (13), and let $\epsilon > 0$ be such that

$$\frac{1}{(f_\infty - \epsilon)K^2 L \int_\xi^\omega G_n(s,s)c(s)\nabla s} \le \lambda \le \frac{1}{(f_0 + \epsilon)L \int_a^b G_n(s,s)c(s)\nabla s}. \tag{14}$$

Since x is a solution of (1), (2) if and only if

$$x(t) = \lambda \int_a^b H_n(t,s)c(s)f(x(s))\nabla s, \quad t \in [a,b],$$

define the operator $T : \mathcal{P} \to \mathcal{B}$ by

$$(Tx)(t) := \lambda \int_a^b H_n(t,s)c(s)f(x(s))\nabla s, \quad x \in \mathcal{P}. \tag{15}$$

We seek a fixed point of T in $\mathcal{P}$ by establishing the hypotheses of Theorem 4.1. First, if $x \in \mathcal{P}$ then by (5) we have

$$\begin{aligned}(Tx)(t) &= \lambda \int_a^b H_n(t,s)c(s)f(x(s))\nabla s \\ &\le \lambda L \int_a^b G_n(s,s)c(s)f(x(s))\nabla s,\end{aligned}$$

so that for $t \in [\xi, \omega]$,

$$\begin{aligned}(Tx)(t) &= \lambda \int_a^b H_n(t,s)c(s)f(x(s))\nabla s \\ &\ge \lambda K L \int_a^b G_n(s,s)c(s)f(x(s))\nabla s \\ &\ge \lambda K \int_a^b H_n(s,s)c(s)f(x(s))\nabla s \\ &\ge K\|Tx\|.\end{aligned}$$

Therefore $T : \mathcal{P} \to \mathcal{P}$. Moreover, T is completely continuous by a typical application of the Ascoli-Arzela Theorem.

Now consider f_0. There exists an $R_1 > 0$ such that $f(x) \le (f_0 + \epsilon)x$ for $0 < x \le R_1$ by the definition of f_0. Pick $x \in \mathcal{P}$ with $\|x\| = R_1$. Using (5) we have

$$\begin{aligned}(Tx)(t) &\le \lambda(f_0 + \epsilon)L\|x\| \int_a^b G_n(s,s)c(s)\nabla s \\ &\le \|x\|\end{aligned}$$

from the right side of (14). As a result, $\|Tx\| \leq \|x\|$. Thus, take

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < R_1\}$$

so that $\|Tx\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_1$.

Next consider f_∞. Again by definition there exists an $R_2' > R_1$ such that $f(x) \geq (f_\infty - \epsilon)x$ for $x \geq R_2'$; take $R_2 = \max\{2R_1, R_2'/K\}$. If $x \in \mathcal{P}$ with $\|x\| = R_2$, then for $t \in [\xi, \omega]$ we have

$$x(t) \geq K\|x\| = KR_2. \tag{16}$$

Define $\Omega_2 := \{x \in \mathcal{B} : \|x\| < R_2\}$; using (16) for $t \in [\xi, \omega]$ we get

$$\begin{aligned}(Tx)(t) &\geq \lambda KL \int_\xi^\omega G_n(s,s)c(s)f(x(s))\nabla s \\ &\geq \lambda KL(f_\infty - \epsilon) \int_\xi^\omega G_n(s,s)c(s)x(s)\nabla s \\ &\geq \lambda(f_\infty - \epsilon)K^2R_2L \int_\xi^\omega G_n(s,s)c(s)\nabla s \\ &\geq R_2 \\ &= \|x\|,\end{aligned}$$

where we have used the left side of (14). Hence we have shown that

$$\|Tx\| \geq \|x\|, \quad x \in \mathcal{P} \cap \partial\Omega_2.$$

An application of Theorem 4.1 yields the conclusion of the theorem. □

Theorem 4.3 *Suppose (A1) and (A2) hold. Then for each λ satisfying*

$$\frac{1}{f_0K^2L\int_\xi^\omega G_n(s,s)c(s)\nabla s} < \lambda < \frac{1}{f_\infty L\int_a^b G_n(s,s)c(s)\nabla s} \tag{17}$$

there exists at least one positive solution of (1), (2) in $\mathcal{P}$.

Proof: Let λ be as in (17) and let $\eta > 0$ be such that

$$\frac{1}{(f_0 - \eta)K^2L\int_\xi^\omega G_n(s,s)c(s)\nabla s} \leq \lambda \leq \frac{1}{(f_\infty + \eta)L\int_a^b G_n(s,s)c(s)\nabla s}. \tag{18}$$

Let T be the completely continuous, cone-preserving operator defined in (15). We seek a fixed point of T in $\mathcal{P}$ by establishing the hypotheses of Theorem 4.1.

First, consider f_0. There exists an $R_1 > 0$ such that $f(x) \geq (f_0 - \eta)x$ for $0 < x \leq R_1$ by the definition of f_0. Pick $x \in \mathcal{P}$ with $\|x\| = R_1$. For $t \in [\xi, \omega]$, where ξ, ω are as in (8), we have

$$x(t) \geq K\|x\| = KR_1. \tag{19}$$

Using the left side of (18) and (19) we get, for $t \in [\xi, \omega]$,

$$\begin{aligned}
(Tx)(t) &\geq \lambda KL \int_\xi^\omega G_n(s,s)c(s)f(x(s))\nabla s \\
&\geq \lambda(f_0 - \eta)KL \int_\xi^\omega G_n(s,s)c(s)x(s)\nabla s \\
&\geq \lambda(f_0 - \eta)R_1K^2L \int_\xi^\omega G_n(s,s)c(s)\nabla s \\
&\geq R_1 \\
&= \|x\|.
\end{aligned}$$

Therefore $\|Tx\| \geq \|x\|$. This prompts us to define

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < R_1\},$$

whereby our work above confirms

$$\|Tx\| \geq \|x\|, \ \ x \in \mathcal{P} \cap \partial\Omega_1.$$

Next consider f_∞. Again by definition there exists an $R_2' > R_1$ such that $f(x) \leq (f_\infty + \eta)x$ for $x \geq R_2'$. If f is bounded, there exists $M > 0$ with $f(x) \leq M$ for all $x \in (0, \infty)$. Let

$$R_2 := \max\{2R_2', \lambda LM \int_a^b G_n(s,s)c(s)\nabla s\}.$$

If $x \in \mathcal{P}$ with $\|x\| = R_2$, then we have

$$\begin{aligned}
(Tx)(t) &= \lambda \int_a^b H_n(t,s)c(s)f(x(s))\nabla s \\
&\leq \lambda L \int_a^b G_n(s,s)c(s)f(x(s))\nabla s \\
&\leq \lambda LM \int_a^b G_n(s,s)c(s)\nabla s \\
&\leq R_2 \\
&= \|x\|.
\end{aligned}$$

As a result, $\|Tx\| \leq \|x\|$. Thus, take

$$\Omega_2 := \{x \in \mathcal{B} : \|x\| < R_2\}$$

so that $\|Tx\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial\Omega_2$. If f is unbounded, take $R_2 := \max\{2R_1, R_2'\}$ such that $f(x) \leq f(R_2)$ for $0 < x \leq R_2$. If $x \in \mathcal{P}$ with $\|x\| = R_2$, then we have

$$\begin{aligned} (Tx)(t) &\leq \lambda L \int_a^b G_n(s,s)c(s)f(R_2)\nabla s \\ &\leq \lambda(f_\infty + \eta)R_2 L \int_a^b G_n(s,s)c(s)\nabla s \\ &\leq R_2 \\ &= \|x\|, \end{aligned}$$

where we have used the left side of (18). Hence we have shown that

$$\|Tx\| \leq \|x\|, \ \ x \in \mathcal{P} \cap \partial\Omega_2$$

if we take

$$\Omega_2 := \{x \in \mathcal{B} : \|x\| < R_2\}.$$

Once again an application of Theorem 4.1 yields the conclusion of the theorem. □

Corollary 4.4 *Suppose* (A1) *and* (A2) *hold. If f is sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$), or if f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$), then for any $\lambda > 0$ the boundary value problem* (1), (2) *has at least one positive solution in $\mathcal{P}$.*

Proof: For the superlinear claim, use (13) of Theorem 4.2; for the sublinear claim, use (17) of Theorem 4.3. □

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