

Existence of Solutions for a One Dimensional p -Laplacian on Time-Scales

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We prove the existence of at least one positive solution to the time-scale, delta–nabla dynamic equation, $(g(u^\Delta))^\nabla + c(t)f(u) = 0$, with boundary conditions, $u(a) - B_0(u^\Delta(\nu)) = 0$ and $u^\Delta(b) = 0$. Here, $g(z) = |z|^{p-2}z$ for $p > 1$, $\nu \in (a, b)$, f and c are left-dense continuous and B_0 is a function “bounded” by two linear rays.

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INTRODUCTION

We study the time-scale, delta–nabla dynamic equation,

$$(g(u^\Delta))^\nabla + c(t)f(u) = 0 \quad \text{for } a < t < b, \quad (1)$$

with boundary conditions,

$$u(a) - B_0(u^\Delta(\nu)) = 0 \quad \text{and} \quad u^\Delta(b) = 0, \quad (2)$$

where $g(z) = |z|^{p-2}z$ for $p > 1$, and $\nu \in (a, b) \subset \mathbb{T}$ for any time-scale \mathbb{T} . Moreover, the functions $f : [0, \infty) \rightarrow [0, \infty)$ and $c : [a, b] \rightarrow [0, \infty)$ are left-dense continuous. Let $m := \int_a^b c(r) \nabla r > 0$, $M := \int_a^b c(r) \nabla r > 0$; we assume that there exist constants $K_m, K_M \in (0, \infty)$ such that

$$K_m x \leq B_0(x) \leq K_M x$$

for all $x \geq 0$. Note, $G(w) = |w|^{1/(p-1)} \text{sgn}(w)$ is the inverse of g . We will refer to $(g(u^\Delta))^\nabla$ as a p -Laplacian operator. In the case of the time-scales \mathbb{R} (the real numbers), or $h\mathbb{Z}$ (a constant graininess), the p -Laplacian arises in non-Newtonian fluids, in some reaction–diffusion problems, in flow through porous media, in nonlinear elasticity, glaciology and petroleum extraction; for a few references to such applications, see Refs. [3,7,8,15,20,22,32,33].

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Our approach will involve an application of a functional-type cone expansion–compression fixed point theorem. Since 1997, a good deal of attention has been given to applications of fixed point theorems yielding positive solutions for certain boundary value problems, with the landmark papers by Erbe and Wang [21] and Wang [34] motivating such of the attention. Applications of the Guo-Krasnosel’skii fixed point theorem [19,23,28,35], the Leggett–Williams fixed point theorem [29], the Avery five functional fixed point theorem [11] and extensions of those fixed point theorems in Ref. [14,24,25] have been made to yield positive solutions, multiple positive solutions and symmetric solutions for boundary value problems including conjugate, focal, Sturm-Liouville and Lidstone boundary conditions. For a good selection of these results, see the book by Agarwal *et al.* [2], as well as the papers [1,10,12,13,18,26,27,30]. The method here relies at its core on the fundamental theorem of Calculus, a result that has been generalized and extended to time-scales; to gain a good overview of the burgeoning area of time-scale research, see the recent books by Bohner and Peterson [16,17]. For more on delta–nabla dynamic equations, first introduced by Atici and Guseinov [9], see [4–6,31]. In the second section, some cone preliminaries are presented along with the statement of a compression–expansion fixed point theorem in terms of nonnegative continuous functionals. Then, in the third section, growth conditions are imposed on f in terms of g , which are sufficient to apply the fixed point theorem and obtain positive solutions of Eqs. (1) and (2).

PRELIMINARIES

In this section, we provide some background material from the theory of cones in Banach spaces.

DEFINITION 1 *Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following two conditions:*

- (i) $x \in P, \lambda \geq 0$ imply $\lambda x \in P$;
- (ii) $x \in P, -x \in P$ imply $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

DEFINITION 2 *An operator is completely continuous if it is continuous and maps bounded sets into precompact sets.*

DEFINITION 3 *A map α is a nonnegative continuous functional on a cone P of a real Banach space E if*

$$\alpha : P \rightarrow [0, \infty)$$

is continuous.

Let α and γ be nonnegative continuous functionals on P . Then, for positive real numbers r and R , we define the following sets:

$$P(\gamma, R) = \{x \in P : \gamma(x) < R\} \tag{3}$$

and

$$P(\gamma, \alpha, r, R) = \{x \in P : r < \alpha(x) \text{ and } \gamma(x) < R\}. \tag{4}$$

THEOREM 4 (FIXED POINT THEOREM OF CONE EXPANSION AND COMPRESSION OF FUNCTIONAL TYPE) *Let P be a cone in a real Banach space E , and let α and γ be nonnegative continuous functionals on P . Assume $P(\gamma, \alpha, r, R)$ as in Eq. (4) is a nonempty bounded subset of P ,*

$$A : \overline{P(\gamma, \alpha, r, R)} \rightarrow P$$

is a completely continuous operator with

$$\inf_{x \in \partial P(\gamma, \alpha, r, R)} \|Ax\| > 0,$$

and

$$\overline{P(\alpha, r)} \subseteq P(\gamma, R)$$

for these sets as in Eq. (3). If one of the two conditions

$\alpha(Ax) \leq r$ for all $x \in \partial P(\alpha, r)$, $\gamma(Ax) \geq R$ for all $x \in \partial P(\gamma, R)$, and for all

$y \in \partial P(\alpha, r)$, $z \in P(\gamma, R)$, $\lambda \geq 1$ and $\mu \in (0, 1]$ the functionals

$$\text{satisfy the properties } \alpha(\lambda y) \geq \lambda \alpha(y), \quad \gamma(\mu z) \leq \mu \gamma(z), \text{ and } \alpha(0) = 0 \tag{H1}$$

or

$\alpha(Ax) \geq r$ for all $x \in \partial P(\alpha, r)$, $\gamma(Ax) \leq R$ for all $x \in \partial P(\gamma, R)$, and for all

$y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$ and $\mu \geq 1$ the functionals satisfy

the properties $\alpha(\lambda y) \leq \lambda \alpha(y)$, $\gamma(\mu z) \geq \mu \gamma(z)$, and $\gamma(0) = 0$

is satisfied, then A has at least one positive fixed point x^* such that

$$r \leq \alpha(x^*) \text{ and } \gamma(x^*) \leq R. \tag{H2}$$

EXISTENCE OF POSITIVE SOLUTIONS

Let $E = C_{rd}[a, b]$ be the Banach space with the sup-norm, $\|u\| = \sup\{|u(s)| : a \leq s \leq b\}$, and define the cone $P \subset E$ by

$$P = \{v \in E : v \text{ is decreasing with } v(b) = 0\}.$$

For $v \in P$ define the operator T by

$$Tv(t) = G \left(\int_t^b c(r) f \left(B_0(v(v)) + \int_a^r v(s) \Delta s \right) \nabla r \right). \tag{5}$$

LEMMA 5 *If v is a fixed point of T , then*

$$u(t) := B_0(v(\nu)) + \int_a^t v(s)\Delta s$$

is a solution of Eqs. (1) and (2).

Proof Suppose v is a fixed point of T and $u(t) := B_0(v(\nu)) + \int_a^t v(s)\Delta s$. Then

$$u^\Delta(t) = v(t) = Tv(t) = G\left(\int_t^b c(r)f\left(B_0(v(\nu)) + \int_a^r v(s)\Delta s\right)\nabla r\right),$$

so that

$$g(u^\Delta(t)) = \int_t^b c(r)f\left(B_0(v(\nu)) + \int_a^r v(s)\Delta s\right)\nabla r.$$

Therefore,

$$(g(u^\Delta))^\nabla(t) = -c(t)f\left(B_0(v(\nu)) + \int_a^t v(s)\Delta s\right) = -c(t)f(u(t)).$$

□

For $\nu \in (a, b)$ define the nonnegative continuous functionals α and γ on P by

$$\gamma(v) := v(\nu)$$

and

$$\alpha(v) := \int_a^\nu v(s)\Delta s.$$

The following two lemmas highlight two essential inequalities that we will apply in our main existence theorem.

LEMMA 6 *If $v \in P$ with $\gamma(v) = R$, then*

$$\int_a^\nu v(s)\Delta s \geq (\nu - a)R.$$

Proof Suppose $\gamma(v) = R$ and $v \in P$. Since v is decreasing on $[a, b]$, for $s \in [a, \nu]$ we have

$$v(s) \geq v(\nu) = \gamma(v) = R.$$

Therefore

$$\int_a^\nu v(s)\Delta s \geq \int_a^\nu R\Delta s = (\nu - a)R.$$

□

LEMMA 7 *If $v \in P$ with $\alpha(v) = r$, then*

$$v(\nu) \leq \frac{r}{\nu - a},$$

and

$$\int_a^b v(s)\Delta s \leq \frac{r(b-a)}{\nu-a}.$$

Proof Suppose $v \in P$ with $\alpha(v) = r$. Since v is decreasing on $[a, b]$,

$$r = \alpha(v) = \int_a^\nu v(s)\Delta s \geq \int_a^\nu v(\nu)\Delta s = (\nu-a)v(\nu).$$

Thus

$$v(\nu) \leq \frac{r}{\nu-a},$$

and

$$\begin{aligned} \int_a^b v(s)\Delta s &= \int_a^\nu v(s)\Delta s + \int_\nu^b v(s)\Delta s \\ &\leq r + \int_\nu^b v(\nu)\Delta s \\ &= r + (b-\nu)v(\nu) \\ &\leq r + \frac{r(b-\nu)}{\nu-a} \\ &= \frac{r(b-a)}{\nu-a}. \end{aligned}$$

□

We now prove our main existence theorem for the time-scale one-dimensional p -Laplacian boundary value problem.

THEOREM 8 *Suppose there exist positive real numbers r and R with $r \leq (\nu-a)R$ and left-dense continuous functions $f : [0, \infty) \rightarrow [0, \infty)$ and $c : [a, b] \rightarrow [0, \infty)$ such that the following conditions are met:*

- (i) $f(w) \geq k$ for some positive constant k , for all $w \in [r, \infty)$,
- (ii) $f(w) \leq \frac{g(\frac{r}{M})}{M}$ for all $w \in [0, \frac{r}{\nu-a}(K_M + b - a)]$,
- (iii) $f(w) \leq \frac{g(R)}{m}$ for all $w \in [R(K_m + \nu - a), \infty)$.

Then the operator T has at least one fixed point v^* such that

$$r \leq \alpha(v^*) \text{ and } \gamma(v^*) \leq R.$$

Proof We will invoke the fixed point theorem of cone compression and expansion of functional type (Theorem 4) once we have shown that the hypotheses (H2) have been satisfied. We have that P is a cone in the Banach space E with the sup norm. From Eq. (5), $T:P \rightarrow P$ since $f(x) \geq 0$ on $[0, \infty)$, $c(t) \geq 0$ on $[a, b]$, G is increasing on $[0, \infty)$, and $\int_t^b c(r)f(B_0(v(\nu)) + \int_a^\nu v(s)\Delta s)\nabla r$ is decreasing for all $v \in P$. For $z \in \partial P(\gamma, R)$ and $\mu \geq 1$,

$$\gamma(\mu z) = \mu z(\nu) = \mu \gamma(z)$$

with $\gamma(0) = 0$; similarly for $y \in \partial P(\gamma, R)$ and $\lambda \in (0, 1]$,

$$\alpha(\lambda y) = \int_a^\nu \lambda y(s)\Delta s = \lambda \int_a^\nu y(s)\Delta s = \lambda \alpha(y).$$

If $u \in \overline{P(\alpha, r)}$, then

$$\gamma(u) = u(\nu) \leq \frac{1}{\nu - a} \int_a^\nu u(s) \Delta s = \frac{1}{\nu - a} \alpha(u) \leq \frac{r}{\nu - a} \leq R,$$

so that $u \in P(\gamma, R)$ and $\overline{P(\alpha, r)} \subseteq P(\gamma, R)$. If $u \in \partial P(\gamma, \alpha, r, R)$, then by (i) we have

$$\begin{aligned} \|Tu\| &= G\left(\int_a^b c(r) f\left(B_0(u(\nu)) + \int_a^r u(s) \Delta s\right) \nabla r\right) \\ &\geq G\left(\int_\nu^b c(r) f\left(B_0(u(\nu)) + \int_a^r u(s) \Delta s\right) \nabla r\right) \\ &\geq G\left(k \int_\nu^b c(r) \nabla r\right), \end{aligned}$$

so that

$$\inf_{u \in \partial P(\gamma, \alpha, r, R)} \|Tu\| \geq G\left(k \int_\nu^b c(r) \nabla r\right) > 0.$$

If $v \in \partial P(\alpha, r)$, then by Lemma 7

$$\int_a^b v(s) \Delta s \leq \frac{r(b-a)}{\nu-a};$$

Hence for $r \in [a, b]$,

$$0 \leq B_0(v(\nu)) + \int_a^r v(s) \Delta s \leq \frac{r}{\nu-a} (K_M + b - a).$$

It follows from condition (ii) that

$$\begin{aligned} \alpha(Tv) &= \int_a^\nu G\left(\int_t^b c(r) f\left(B_0(v(\nu)) + \int_a^r v(s) \Delta s\right) \nabla r\right) \Delta t \\ &\leq \int_a^\nu G\left(\int_a^b c(r) f\left(B_0(v(\nu)) + \int_a^r v(s) \Delta s\right) \nabla r\right) \Delta t \\ &\leq (\nu - a) G\left(\left(\frac{g\left(\frac{r}{\nu-a}\right)}{M}\right) \int_a^b c(r) \nabla r\right) = r. \end{aligned}$$

Therefore, $\alpha(Tv) \geq r$. If $v \in \partial P(\gamma, R)$, then by Lemma 6 we have

$$\int_a^\nu v(s) \Delta s \geq (\nu - a)R;$$

Hence, for $r \in [\nu, b]$,

$$K_m R + (\nu - a)R \leq B_0(v(\nu)) + \int_a^\nu v(s) \Delta s \leq B_0(v(\nu)) + \int_a^r v(s) \Delta s.$$

Thus, by condition (iii),

$$\begin{aligned} \gamma(Tv) &= G\left(\int_{\nu}^b c(r)f\left(B_0(v(\nu)) + \int_a^r v(s)\Delta s\right)\nabla r\right) \\ &\leq G\left(\left(\frac{g(R)}{m}\right)\int_{\nu}^b c(r)\nabla r\right) = R. \end{aligned}$$

Therefore, $\gamma(Tv) \leq R$. Therefore, the hypotheses of the fixed point theorem of cone compression and expansion of functional type are satisfied. Thus, the operator T has a fixed point v^* such that

$$r \leq \alpha(v^*) \text{ and } \gamma(v^*) \leq R.$$

□

EXAMPLE

Let $\mathbb{T} = \mathbb{R}$, and consider the unit interval differential equation

$$(u'|u'|)^{\prime} + 4t \arctan(u) = 0 \text{ for } 0 < t < 1, \tag{6}$$

with boundary conditions

$$u(0) = u'(0.5) + \sin(u'(0.5)) \text{ and } u'(1) = 0. \tag{7}$$

Note that we have taken $p = 3, c(t) = 2t, f(w) = 2 \arctan w, \nu = 0.5$ and $B_0(x) = x + \sin x$ then $m = \int_{\nu}^1 c(t) dt = 0.75, M = \int_0^1 c(t) dt = 1, K_m = 0.5$ and $K_M = 2$ so that

$$K_m x \leq B_0(x) \leq K_M x$$

for all $x \geq 0$. Let $r = 1$ and $R = 3$. Then $r \leq \nu R$, and $f(w) := 2 \arctan w$ maps $[0, \infty)$ to $[0, \infty)$, meeting the following conditions:

- (i) $f(w) \geq 1$ for all $w \in [1, \infty)$,
- (ii) $f(w) \leq 4$ for all $w \in [0, 6]$,
- (iii) $f(w) \leq 12$ for all $w \in [3, \infty)$.

Therefore, by Theorem 8, the operator T is given by

$$Tv(t) = G\left(4\int_t^1 r \arctan\left(v(0.5) + \sin v(0.5) + \int_0^r v(s) ds\right) dr\right)$$

with $G(w) = \text{sgn}(w)\sqrt{|w|}$ has a fixed point v^* such that

$$1 \leq \int_0^{0.5} v^*(s) ds, \quad v^*(0.5) \leq 3.$$

Then

$$u(t) := v^*(0.5) + \sin v^*(0.5) + \int_0^t v^*(s) ds$$

is a solution of the boundary value problem (6) and (7) by Lemma 5.

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