TAYLOR POLYNOMIALS FOR NABLA DYNAMIC EQUATIONS ON TIME SCALES

DOUGLAS R. ANDERSON

Abstract. We are concerned with the representation of polynomials for nabla dynamic equations on time scales. Once established, these polynomials will be used to derive Taylor's Formula for dynamic functions. Several examples are given for special time scales such as $\mathbb{Z}$, $\mathbb{R}$, and $\overline{q^{\mathbb{Z}}}$ for $q > 1$. These polynomials will be related to those given for delta dynamic equations.

1. Preliminaries on Time Scales

The following definitions can be found in Agarwal and Bohner [1], Atici and Guseinov [2], and Bohner and Peterson [3]; these are based on the notions first introduced by Hilger in his Ph.D. thesis [4]. A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\nu(t) := t - \rho(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [3] of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, we have $f^\Delta = f'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have the forward difference operator, $f^\Delta(t) = f(t+1) - f(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [2] of f at t, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a

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A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous. It is known [3] that if f is ld-continuous, then there is a function $F(t)$ such that $F^{\nabla}(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

Remark 1. The following theorems delineate several properties of the delta and nabla derivatives; they are found in [2] and [3, p331-333].

Theorem 1. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

(i) *If f is delta differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is delta differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is delta differentiable at t iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is delta differentiable at t, then*

$$f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t),$$

where $f^{\sigma} = f \circ \sigma$.

Theorem 2. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_{\kappa}$. Then we have the following:*

(i) *If f is nabla differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with*

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

(iii) *If t is left-dense, then f is nabla differentiable at t iff the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case

$$f^{\nabla}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iv) *If f is nabla differentiable at t, then*

$$f^{\rho}(t)=f(t)-\nu(t)f^{\nabla}(t),$$

where $f^{\rho}=f\circ\rho$.

Theorem 3. *Assume $f,g:\mathbb{T}\to\mathbb{R}$ are nabla differentiable at $t\in\mathbb{T}_{\kappa}$. Then:*

(i) *The sum $f+g:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t with*

$$(f+g)^{\nabla}(t)=f^{\nabla}(t)+g^{\nabla}(t).$$

(ii) *The product $fg:\mathbb{T}\to\mathbb{R}$ is nabla differentiable at t, and we get the product rules*

$$(fg)^{\nabla}(t)=f^{\nabla}(t)g(t)+f^{\rho}(t)g^{\nabla}(t)=f(t)g^{\nabla}(t)+f^{\nabla}(t)g^{\rho}(t).$$

(iii) *If $g(t)g^{\rho}(t)\neq 0$, then $\frac{f}{g}$ is nabla differentiable at t, and we get the quotient rule*

$$\left(\frac{f}{g}\right)^{\nabla}(t)=\frac{f^{\nabla}(t)g(t)-f(t)g^{\nabla}(t)}{g(t)g^{\rho}(t)}.$$

(iv) *If f and f^{∇} are continuous, then*

$$\left(\int_a^t f(t,s)\nabla s\right)^{\nabla}=f(\rho(t),t)+\int_a^t f^{\nabla}(t,s)\nabla s.$$

2. Monomials

The generalized monomials, that will also occur in Taylor's formula, are the functions $\hat{h}_k:\mathbb{T}^2\to\mathbb{R}$, $k\in\mathbb{N}_0$, defined recursively as follows: The function $\hat{h}_0$ is

$$\hat{h}_0(t,s)\equiv 1 \quad \text{for all} \quad s,t\in\mathbb{T}, \tag{1}$$

and, given $\hat{h}_k$ for $k\in\mathbb{N}_0$, the function $\hat{h}_{k+1}$ is

$$\hat{h}_{k+1}(t,s)=\int_s^t \hat{h}_k(\tau,s)\nabla\tau \quad \text{for all} \quad s,t\in\mathbb{T}. \tag{2}$$

Note that the functions $\hat{h}_k$ are all well defined, since each is ld-continuous. If we let $\hat{h}_k^{\nabla}(t,s)$ denote for each fixed s the derivative of $\hat{h}_k(t,s)$ with respect to t, then

$$\hat{h}_k^{\nabla}(t,s)=\hat{h}_{k-1}(t,s) \quad \text{for} \quad k\in\mathbb{N},\ t\in\mathbb{T}_{\kappa}. \tag{3}$$

The above definition implies

$$\hat{h}_1(t,s) = t - s \quad \text{for all} \quad s,t \in \mathbb{T}.$$

Finding the $\hat{h}_k$ for $k > 1$ is not easy in general. But for a particular given time scale it might be easy to find these functions. We will consider several examples first before we present Taylor's formula for arbitrary time scales.

Example 1. For the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ it is easy to find the functions $\hat{h}_k$:

First, consider $\mathbb{T} = \mathbb{R}$. Then $\rho(t) = t$ for $t \in \mathbb{R}$, so that

$$\hat{h}_k(t,s) = \frac{(t-s)^k}{k!}$$

for $s,t \in \mathbb{R}$, $k \in \mathbb{N}_0$. We note that, for an n times differentiable function $f : \mathbb{R} \to \mathbb{R}$, the following well-known Taylor's formula holds: Let $\alpha \in \mathbb{R}$ be arbitrary. Then, for all $t \in \mathbb{R}$, the representations

$$\begin{aligned} f(t) &= \sum_{k=0}^{n} \frac{(t-\alpha)^k}{k!} f^{(k)}(\alpha) + \frac{1}{n!}\int_\alpha^t (t-\tau)^n f^{(n+1)}(\tau)d\tau \\ &= \sum_{k=0}^{n} \hat{h}_k(t,\alpha) f^{(k)}(\alpha) + \int_\alpha^t \hat{h}_n(t,\rho(\tau)) f^{(n+1)}(\tau)d\tau \end{aligned} \tag{4}$$

are valid, where $f^{(k)}$ denotes as usual the kth derivative of f.

Next, consider $\mathbb{T} = \mathbb{Z}$. Then $\rho(t) = t - 1$, $\nabla f(t) = f(t) - f(t-1)$ for $t \in \mathbb{Z}$, and [3, p333]

$$\int_a^b f(t)\nabla t = \sum_{t=a+1}^{b} f(t).$$

For a nonnegative integer k and $t \in \mathbb{Z}$, define the discrete factorial function t to the k rising by $t^{\overline{0}} := 1$ for $k = 0$, and

$$t^{\overline{k}} := t(t+1)\cdots(t+k-1)$$

for $k \geq 1$; then

$$\nabla t^{\overline{k}} = k t^{\overline{k-1}}.$$

For $s,t \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ we have

$$\hat{h}_k(t,s) = \frac{(t-s)^{\overline{k}}}{k!} \quad \text{for all} \quad s,t \in \mathbb{Z}. \tag{5}$$

Therefore the discrete nabla version of Taylor's formula reads as follows: Let $f : \mathbb{Z} \to \mathbb{R}$ be a function, and let $\alpha \in \mathbb{Z}$. Then, for all $t \in \mathbb{Z}$ with $t > \alpha + n$, the representations

$$\begin{aligned} f(t) &= \sum_{k=0}^{n} \frac{(t-\alpha)^{\overline{k}}}{k!} \nabla^k f(\alpha) + \frac{1}{n!} \sum_{\tau=\alpha+1}^{t} (t-\tau+1)^{\overline{n}} \nabla^{n+1} f(\tau) \\ &= \sum_{k=0}^{n} \hat{h}_k(t,\alpha) \nabla^k f(\alpha) + \sum_{\tau=\alpha+1}^{t} \hat{h}_n(t,\rho(\tau)) \nabla^{n+1} f(\tau) \end{aligned} \tag{6}$$

hold, where ∇^k is the k-times iterated backward difference operator.

Example 2. We consider the time scale

$$\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{0, 1, q, q^{-1}, q^2, q^{-2}, \ldots\}$$

for some $q > 1$. Here $\nu(t) = t - \rho(t) = t - t/q = (q-1)t/q$. From [3, Example 1.104] and Theorem 9 later in this paper we have that

$$\hat{h}_k(t,s) = \prod_{r=0}^{k-1} \frac{q^r t - s}{\sum_{j=0}^{r} q^j} \quad \text{for all} \quad s, t \in \mathbb{T} \tag{7}$$

holds for all $k \in \mathbb{N}_0$.

3. Taylor's Theorem

We will consider Taylor's Theorem in the context of higher-order dynamic equations. In this setting, the theorem will be proved using the variation of constants formula. With this in mind, we begin this section with a general higher-order equation. For $n \in \mathbb{N}_0$ and ld-continuous functions $p_i : \mathbb{T} \to \mathbb{R}$, $1 \le i \le n$, with regressivity condition

$$1 + \sum_{i=1}^{n} (\nu(t))^i p_i(t) \neq 0 \quad t \in \mathbb{T}_\kappa,$$

we consider the nth order linear dynamic equation

$$Ly = 0, \quad \text{where} \quad Ly = y^{\nabla^n} + \sum_{i=1}^{n} p_i y^{\nabla^{n-i}}. \tag{8}$$

A function $y : \mathbb{T} \to \mathbb{R}$ is a solution of equation (8) on $\mathbb{T}$ provided y is n times nabla differentiable on $\mathbb{T}_{\kappa^n}$ and satisfies $Ly(t) = 0$ for all $t \in \mathbb{T}_{\kappa^n}$. It follows that y^{∇^n} is an ld-continuous function on $\mathbb{T}_{\kappa^n}$.

Now let $f : \mathbb{T} \to \mathbb{R}$ be ld-continuous and consider the *nonhomogeneous* equation

$$y^{\nabla^n}(t) + \sum_{i=1}^{n} p_i(t) y^{\nabla^{n-i}}(t) = f(t). \tag{9}$$

Definition 1. We define the Cauchy function $y : \mathbb{T} \times \mathbb{T}_{\kappa^n} \to \mathbb{R}$ for the linear dynamic equation (8) to be for each fixed $s \in \mathbb{T}_{\kappa^n}$ the solution of the initial value problem

$$Ly = 0, \quad y^{\nabla^i}(\rho(s), s) = 0, \quad 0 \le i \le n-2, \quad y^{\nabla^{n-1}}(\rho(s), s) = 1.$$

Remark 2. Note that

$$y(t,s) := \hat{h}_{n-1}(t, \rho(s))$$

is the Cauchy function for $y^{\nabla^n} = 0$.

Theorem 4 (Variation of Constants)**.** *Let $\alpha \in \mathbb{T}_{\kappa^n}$ and $t \in \mathbb{T}$. If $f \in \mathrm{C}_{\mathrm{ld}}$, then the solution of the initial value problem*

$$Ly = f(t), \quad y^{\nabla^i}(\alpha) = 0, \quad 0 \le i \le n-1$$

is given by

$$y(t) = \int_\alpha^t y(t,\tau) f(\tau) \nabla\tau,$$

where $y(t,\tau)$ is the Cauchy function for (8).

Proof. With y defined as above and by the properties of the Cauchy function we have

$$y^{\nabla^i}(t) = \int_\alpha^t y^{\nabla^i}(t,\tau) f(\tau)\nabla\tau + y^{\nabla^{i-1}}(\rho(t), t) f(t) = \int_\alpha^t y^{\nabla^i}(t,\tau) f(\tau) \nabla\tau$$

for $0 \le i \le n-1$ and

$$y^{\nabla^n}(t) = \int_\alpha^t y^{\nabla^n}(t,\tau) f(\tau)\nabla\tau + y^{\nabla^{n-1}}(\rho(t), t) f(t) = \int_\alpha^t y^{\nabla^n}(t,\tau) f(\tau)\nabla\tau + f(t).$$

It follows from these equations that

$$y^{\nabla^i}(\alpha) = 0, \quad 0 \le i \le n-1$$

and

$$Ly(t) = \int_\alpha^t Ly(t,\tau) f(\tau)\nabla\tau + f(t) = f(t),$$

and the proof is complete. $\square$

Theorem 5 (Taylor's Formula)**.** *Let $n \in \mathbb{N}$. Suppose f is $n+1$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Let $\alpha \in \mathbb{T}_{\kappa^n}$, $t \in \mathbb{T}$, and define the functions $\hat{h}_k$ by* (1) *and* (2)*, i.e.,*

$$\hat{h}_0(r,s) \equiv 1 \quad \text{and} \quad \hat{h}_{k+1}(r,s) = \int_s^r \hat{h}_k(\tau, s)\nabla\tau \text{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n} \hat{h}_k(t,\alpha) f^{\nabla^k}(\alpha) + \int_\alpha^t \hat{h}_n(t, \rho(\tau)) f^{\nabla^{n+1}}(\tau)\nabla\tau.$$

Proof. Let $g(t) := f^{\nabla^{n+1}}(t)$. Then f solves the initial value problem

$$x^{\nabla^{n+1}} = g, \quad x^{\nabla^k}(\alpha) = f^{\nabla^k}(\alpha), \quad 0 \le k \le n.$$

Note that the Cauchy function for $y^{\nabla^{n+1}} = 0$ is $y(t,\alpha) = \hat{h}_n(t, \rho(\alpha))$. By the variation of constants formula,

$$f(t) = u(t) + \int_\alpha^t \hat{h}_n(t, \rho(\tau)) g(\tau) \nabla \tau,$$

where u solves the initial value problem

$$u^{\nabla^{n+1}} = 0, \quad u^{\nabla^m}(\alpha) = f^{\nabla^m}(\alpha), \quad 0 \le m \le n. \tag{10}$$

To validate the claim that $u(t) = \sum_{k=0}^n \hat{h}_k(t,\alpha) f^{\nabla^k}(\alpha)$, set

$$w(t) := \sum_{k=0}^n \hat{h}_k(t,\alpha) f^{\nabla^k}(\alpha).$$

By the properties of the $\hat{h}_k$ given in (1) and (3), $w^{\nabla^{n+1}} = 0$. We have moreover that

$$w^{\nabla^m}(t) = \sum_{k=m}^n \hat{h}_{k-m}(t,\alpha) f^{\nabla^k}(\alpha),$$

so that

$$w^{\nabla^m}(\alpha) = \sum_{k=m}^n \hat{h}_{k-m}(\alpha,\alpha) f^{\nabla^k}(\alpha) = f^{\nabla^m}(\alpha)$$

for $0 \le m \le n$. We consequently have that w also solves (10), whence $u \equiv w$ by uniqueness. □

Remark 3. The reader may compare Example 1 (i.e., the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$) to the above presented theory. Taylor's formula corresponds to formulas (4) and (6).

Corollary 6. *Let $\alpha, \beta \in \mathbb{T}$ be fixed. For any $t \in \mathbb{T}$ and any positive integer n,*

$$\hat{h}_n(t,\beta) = \sum_{k=0}^n \hat{h}_k(t,\alpha) \hat{h}_{n-k}(\alpha,\beta).$$

Proof. Take $f = \hat{h}_n(\cdot, \beta)$ in Taylor's formula. □

Example 3. For $\mathbb{T} = \mathbb{Z}$, consider $f(t) = (\frac{1}{2})^t$ for $t \in \mathbb{Z}$. If we expand f about 0, then Taylor's formula (6) for f is given by

$$\left(\frac{1}{2}\right)^t = P_n(t) + E_n(t),$$

where the Taylor polynomial P_n is defined by

$$P_n(t) := \sum_{k=0}^{n} \frac{(-1)^k t^{\overline{k}}}{k!}$$

and the error term E_n is given by

$$E_n(t) = \begin{cases} \frac{(-1)^{n+1}}{n!} \sum_{\tau=1}^{t} (t+1-\tau)^{\overline{n}} \left(\frac{1}{2}\right)^{\tau} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \le 0 \end{cases}$$

For $t = 1$, $P_n(1) = \frac{1-(-1)^{n+1}}{2}$ and $E_n(1) = \frac{(-1)^{n+1}}{2}$, so that the Taylor polynomial P_n will not converge to f at 1 as $n \to \infty$. If $t \le 0$, however, then

$$\left(\frac{1}{2}\right)^{t} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{\overline{k}}}{k!} = \sum_{k=0}^{|t|} \frac{(-1)^k t^{\overline{k}}}{k!} = P_{|t|}(t) = \sum_{k=0}^{|t|} \binom{|t|}{k}$$

converges easily since t is nonpositive.

4. Further Properties

In this final section we relate the functions $\hat{h}_k$ as introduced in (1) and (2) (which we repeat below) to the functions h_k and g_k in the delta case [1, 3], and the functions $\hat{g}_k$.

Definition 2. For $t, s \in \mathbb{T}$ define the functions

$$h_0(t,s) = g_0(t,s) = \hat{h}_0(t,s) = \hat{g}_0(t,s) \equiv 1,$$

and given $h_n, g_n, \hat{h}_n, \hat{g}_n$ for $n \in \mathbb{N}_0$,

$$h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau$$

$$g_{n+1}(t,s) = \int_s^t g_n(\sigma(\tau),s)\Delta\tau$$

$$\hat{h}_{n+1}(t,s) = \int_s^t \hat{h}_n(\tau,s)\nabla\tau$$

$$\hat{g}_{n+1}(t,s) = \int_s^t \hat{g}_n(\rho(\tau),s)\nabla\tau.$$

We will need the following theorems.

Theorem 7. [3, Theorem 1.112] *The functions g_k and h_k defined above satisfy*

$$h_n(t,s) = (-1)^n g_n(s,t)$$

for all $t \in \mathbb{T}$ and all $s \in \mathbb{T}^{\kappa^n}$.

Theorem 8. [2, Theorems 2.5, 2.6] *If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$ and if f^Δ is continuous on $\mathbb{T}^\kappa$, then f is nabla differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = f^\Delta(\rho(t))$$

for all $t \in \mathbb{T}_\kappa$.

If $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$ and if f^∇ is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable on $\mathbb{T}^\kappa$ and

$$f^\Delta(t) = f^\nabla(\sigma(t))$$

for all $t \in \mathbb{T}^\kappa$.

Theorem 9. *Let $t \in \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$ and $s \in \mathbb{T}^{\kappa^n}$. Then $\hat{h}_n(t,s) = (-1)^n h_n(s,t)$ for all $n \geq 0$. In other words,*

$$\hat{h}_n(t,s) = g_n(t,s), \quad h_n(t,s) = \hat{g}_n(t,s).$$

Proof. For $n = 0$,

$$h_0(t,s) \equiv 1 \equiv (-1)^0 \hat{h}_0(s,t).$$

Assume

$$h_{n-1}(t,s) = (-1)^{n-1}\hat{h}_{n-1}(s,t),$$

$n \geq 0$. Note that since $h_{n-1}(\cdot, s)$ is nabla differentiable, it is continuous for each fixed $s \in \mathbb{T}$. Then, using Theorems 7 and 8

$$\begin{aligned}
\hat{h}_n^\Delta(t,s) &= \hat{h}_n^{\nabla\sigma}(t,s) \\
&= \hat{h}_{n-1}(\sigma(t),s) \\
&= (-1)^{n-1}h_{n-1}(s,\sigma(t)) \\
&= g_{n-1}(\sigma(t),s) \\
&= g_n^\Delta(t,s).
\end{aligned}$$

Therefore

$$\hat{h}_n(t,s) = g_n(t,s) + c(s)$$

for all $t \in \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$ and $s \in \mathbb{T}^{\kappa^n}$. But $\hat{h}_n(s,s) = 0 = g_n(s,s)$, so that $c(s) \equiv 0$. It follows that

$$\hat{h}_n(t,s) = g_n(t,s) = (-1)^n h(s,t)$$

and the induction is complete. The proof relating h_n and $\hat{g}_n$ is similar. $\square$

Remark 4. Recall that

$$f^\rho = f - \nu f^\nabla,$$

so that

$$\hat{h}_n(\rho(t),s) = \hat{h}_n(t,s) - \nu(t)\hat{h}_{n-1}(t,s)$$

for $t \in \mathbb{T}_\kappa$ and $s \in \mathbb{T}$.

Lemma 10. *Let $n \in \mathbb{N}_0$. For $t \in \mathbb{T}$ and $s \in \mathbb{T}_\kappa$,*

$$\hat{h}_n(t, \rho(s)) = \sum_{k=0}^{n} \nu^k(s) \hat{h}_{n-k}(t, s).$$

Proof. For $n = 0$,

$$\hat{h}_0(t, \rho(s)) \equiv 1 \equiv \sum_{k=0}^{0} \nu^k(s) \hat{h}_{0-k}(t, s).$$

Assume

$$\hat{h}_n(t, \rho(s)) = \sum_{k=0}^{n} \nu^k(s) \hat{h}_{n-k}(t, s).$$

Then

$$\begin{aligned}
\hat{h}_{n+1}(t, \rho(s)) &= \int_{\rho(s)}^{t} \hat{h}_n(\tau, \rho(s)) \nabla\tau \\
&= \left(\int_s^t + \int_{\rho(s)}^{s} \right) \sum_{k=0}^{n} \nu^k(s) \hat{h}_{n-k}(\tau, s) \nabla\tau \\
&= \sum_{k=0}^{n} \nu^k(s) \int_s^t \hat{h}_{n-k}(\tau, s) \nabla\tau \\
&\quad + \nu(s) \sum_{k=0}^{n} \nu^k(s) \hat{h}_{n-k}(s, s) \\
&= \sum_{k=0}^{n} \nu^k(s) \hat{h}_{n+1-k}(t, s) + \nu^{n+1}(s) \\
&= \sum_{k=0}^{n+1} \nu^k(s) \hat{h}_{n+1-k}(t, s),
\end{aligned}$$

which completes the induction. □

Corollary 11. *Fix $c \in \mathbb{T}^{\kappa^n}$ and $n \in \mathbb{N}_0$. Then*

$$\hat{h}_{n+1}^{\nabla}(c, t) = -\sum_{k=0}^{n} \nu^k(t) \hat{h}_{n-k}(c, t)$$

for $t \in \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

Proof. Let $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
\hat{h}_{n+1}^{\nabla}(c,t) &= (-1)^{n+1} h_{n+1}^{\nabla}(t,c) \\
&= (-1)^{n+1} h_{n+1}^{\Delta\rho}(t,c) \\
&= (-1)^{n+1} h_n(\rho(t),c) \\
&= (-1)^{n+1}(-1)^n \hat{h}_n(c,\rho(t)) \\
&= -\sum_{k=0}^{n} \nu^k(t)\hat{h}_{n-k}(c,t),
\end{aligned}$$

where we have used Theorems 8 and 9. □

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Concordia College, Department of Mathematics and Computer Science, Moorhead, MN 56562 USA

E-mail address: andersod@cord.edu