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Eigenvalue intervals for a two-point boundary value problem on a measure chain

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Abstract

We study the existence of eigenvalue intervals for the second-order differential equation on a measure chain, $x^{\Delta\Delta}(t) + \lambda p(t) f(x^{\sigma}(t)) = 0$, $t \in [t_1, t_2]$, satisfying the boundary conditions $\alpha x(t_1) - \beta x^{\Delta}(t_1) = 0$ and $\gamma x(\sigma(t_2)) + \delta x^{\Delta}(\sigma(t_2)) = 0$, where f is a positive function and p a nonnegative function that is allowed to vanish on some subintervals of $[t_1, \sigma(t_2)]$ of the measure chain. The methods involve applications of a fixed point theorem for operators on a cone in a Banach space.

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1. Introduction

One goal as the result of Hilger's [18] initial paper introducing measure chains has been the unification of the continuous and discrete calculus, and then extending those results to differential equations on time scales. Some other early papers in this area include Agarwal and Bohner [1], Aulbach and Hilger [6] and Erbe and Hilger [12].

One particular area receiving current attention is the question of obtaining optimal eigenvalue intervals of boundary value problems for ordinary differential equations, as well as for finite difference equations. Many of these works have used Krasnoselskii [21] fixed point theorems to obtain intervals based on positive solutions inside a cone. A few papers along these lines are in [3–5, 9,11,16,17,19,20].

Naturally many of these methods carry over when seeking eigenintervals of boundary value problems for differential equations on measure chains; see [2,7,8,13–15], for example.

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In this paper, we are concerned with the existence of eigenvalues for a second-order differential equation on a measure chain satisfying Sturm–Liouville-like boundary conditions. Section 2 introduces this boundary value problem.

2. Eigenvalue intervals

Consider the second-order conjugate boundary value problem

$$-x^{\Delta\Delta}(t) = \lambda p(t) f(x^{\sigma}(t)), \tag{1}$$

$$\begin{aligned} &\alpha x(t_1) - \beta x^{\Delta}(t_1) = 0, \\ &\gamma x(\sigma(t_2)) + \delta x^{\Delta}(\sigma(t_2)) = 0, \end{aligned} \tag{2}$$

where $t_1, t_2 \in \mathbb{T}$, a measure chain, with $t_1 < t_2$. The Green's function [13] for the related homogeneous problem $-x^{\Delta\Delta}(t) = 0$ with boundary conditions (2) is given by

$$G(t,s) = \begin{cases} \frac{1}{d}\{\alpha(t - t_1) + \beta\}\{\gamma(\sigma(t_2) - \sigma(s)) + \delta\}, & t \leqslant s, \\ \frac{1}{d}\{\alpha(\sigma(s) - t_1) + \beta\}\{\gamma(\sigma(t_2) - t) + \delta\}, & \sigma(s) \leqslant t, \end{cases} \tag{3}$$

where $\alpha, \beta, \gamma, \delta \geqslant 0$ and

$$d := \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(t_2) - t_1) > 0.$$

For the rest of this section we have the assumptions [9,20]

(A1) $p(t)$ is a nonnegative, right-dense continuous function defined on $[t_1, \sigma(t_2)]$ satisfying

$$0 < \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s)\, \Delta s < \infty, \tag{4}$$

where, using (3),

$$G(\sigma(s), s) = \frac{1}{d}[\alpha(\sigma(s) - t_1) + \beta][\gamma(\sigma(t_2) - \sigma(s)) + \delta].$$

(A2) $f : [0, \infty) \to [0, \infty)$ is continuous such that both

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{and} \quad f_\infty := \lim_{x \to \infty} \frac{f(x)}{x}$$

exist.

Lemma 1. *For all* $s \in [t_1, t_2]$ *and* $t \in [t_1, \sigma(t_2)]$,

$$g(t) G(\sigma(s), s) \leqslant G(t, s) \leqslant G(\sigma(s), s), \tag{5}$$

where

$$g(t) := \min\left\{ \frac{\alpha(t - t_1) + \beta}{\alpha(\sigma(t_2) - t_1) + \beta}, \frac{\gamma(\sigma(t_2) - t) + \delta}{\gamma(\sigma(t_2) - \sigma(t_1)) + \delta} \right\}. \tag{6}$$

Proof. We have from [13] that

$$\frac{G(t,s)}{G(\sigma(s),s)} = \begin{cases} \dfrac{\alpha(t-t_1)+\beta}{\alpha(\sigma(s)-t_1)+\beta}, & t \leqslant s, \\ \dfrac{\gamma(\sigma(t_2)-t)+\delta}{\gamma(\sigma(t_2)-\sigma(s))+\delta}, & \sigma(s) \leqslant t, \end{cases}$$

which implies the result. □

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnoselskii [21], which can also be found in the book in [10].

Theorem 2. *Let E be a Banach space, $K \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Suppose further that $A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator such that either*
(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$
holds. Then A has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Let $\mathscr{B}$ denote the Banach space $C_{rd}[t_1, \sigma(t_2)]$ with the norm $\|x\| = \sup_{t \in [t_1, \sigma(t_2)]} |x(t)|$. Define the cone $\mathscr{P} \subset \mathscr{B}$ by

$$\mathscr{P} = \{x \in \mathscr{B} \colon x(t) \geqslant g(t)\|x\|, t \in [t_1, \sigma(t_2)]\},$$

where g is given in (6). Let $t_1 < \xi < \omega < \sigma(t_2)$ be chosen from $\mathbb{T}$ such that

$$\int_\xi^\omega G(\sigma(s),s)p(s)\,\Delta s > 0, \tag{7}$$

as p is a nonnegative function, this allows p to vanish on some subintervals. For ease of notation in the following discussion, set

$$k := \min_{t \in [\xi, \sigma(\omega)]} g(t) \tag{8}$$

for g as in (6). Moreover, let K and $\tau \in [t_1, \sigma(t_2)]$ be defined by

$$K := g(\tau) = \max_{t \in [t_1, \sigma(t_2)]} g(t). \tag{9}$$

Theorem 3. *Suppose* (A1) *and* (A2) *hold. Then for each λ satisfying*

$$\frac{1}{f_\infty kK \int_\xi^\omega G(\sigma(s),s)p(s)\,\Delta s} < \lambda < \frac{1}{f_0 \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)\,\Delta s} \tag{10}$$

there exists at least one solution of (1) *and* (2) *in $\mathscr{P}$, for ξ, ω as in* (7), *k as in* (8) *and K as in* (9).

Proof. Let k be as in (8), K as in (9), λ as in (10), and let $\varepsilon > 0$ be such that

$$\frac{1}{(f_\infty - \varepsilon)kK\int_\xi^\omega G(\sigma(s),s)p(s)\,\Delta s} \leqslant \lambda \leqslant \frac{1}{(f_0+\varepsilon)\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)\,\Delta s}. \tag{11}$$

Since $x(t)$ is a solution of (1) and (2) if and only if

$$x(t) = \lambda\int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s, \quad t\in[t_1,\sigma(t_2)],$$

we define the operator $T:\mathscr{P}\to\mathscr{B}$ by

$$(Tx)(t) := \lambda\int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s, \quad x\in\mathscr{P}. \tag{12}$$

We seek a fixed point of T in $\mathscr{P}$ by establishing the hypotheses of Theorem 2. First, if $x\in\mathscr{P}$ then by (5) we have

$$\begin{aligned}(Tx)(t) &= \lambda\int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s\\ &\leqslant \lambda\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)f(x^\sigma(s))\,\Delta s,\end{aligned}$$

so that

$$\begin{aligned}(Tx)(t) &= \lambda\int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s\\ &\geqslant \lambda g(t)\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)f(x^\sigma(s))\,\Delta s\\ &\geqslant g(t)\|Tx\|.\end{aligned}$$

Therefore $T:\mathscr{P}\to\mathscr{P}$. Moreover, T is completely continuous by a typical application of the Ascoli–Arzela Theorem.

Now consider f_0. There exists an $H_1 > 0$ such that $f(x) \leqslant (f_0+\varepsilon)x$ for $0 < x \leqslant H_1$ by the definition of f_0. Pick $x\in\mathscr{P}$ with $\|x\| = H_1$. Using (5) we have

$$\begin{aligned}(Tx)(t) &= \lambda\int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s\\ &\leqslant \lambda(f_0+\varepsilon)\|x\|\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)\,\Delta s\\ &\leqslant \|x\|\end{aligned}$$

from the right-hand side of (11). As a result, $\|Tx\|\leqslant\|x\|$. Thus, take

$$\Omega_1 := \{x\in\mathscr{B}\colon \|x\| < H_1\}$$

so that $\|Tx\|\leqslant\|x\|$ for $x\in\mathscr{P}\cap\partial\Omega_1$.

Next consider f_∞. Again by definition there exists an $H_2 > H_1$ such that $f(x) \geqslant (f_\infty - \varepsilon)x$ for $x \geqslant H_2$. If $x \in \mathscr{P}$ with $\|x\| = H_2$, then for $t \in [\xi, \sigma(\omega)]$, where ξ, ω is as in (7) and k as in (8), we have

$$x(t) \geqslant g(t)\|x\| = g(t)H_2 \geqslant kH_2. \tag{13}$$

Define $\Omega_2 := \{x \in \mathscr{B}: \|x\| < H_2\}$; using (13) for τ as in (9), we get

$$\begin{aligned}(Tx)(\tau) &= \lambda \int_{t_1}^{\sigma(t_2)} G(\tau, s)p(s)f(x^\sigma(s))\,\Delta s\\ &\geqslant \lambda \int_{\xi}^{\omega} G(\tau, s)p(s)f(x^\sigma(s))\,\Delta s\\ &\geqslant \lambda g(\tau)(f_\infty - \varepsilon) \int_{\xi}^{\omega} G(\sigma(s), s)p(s)x^\sigma(s)\,\Delta s\\ &\geqslant \lambda(f_\infty - \varepsilon)kKH_2 \int_{\xi}^{\omega} G(\sigma(s), s)p(s)\,\Delta s\\ &\geqslant H_2\\ &= \|x\|,\end{aligned}$$

where the penultimate line follows from the left-hand side of (11). Hence, we have shown that

$$\|Tx\| \geqslant \|x\|, \quad x \in \mathscr{P} \cap \partial\Omega_2.$$

An application of Theorem 2 validates the conclusion of the theorem. □

Theorem 4. *Suppose* (A1) *and* (A2) *hold. Then for each λ satisfying*

$$\frac{1}{f_0 kK \int_{\xi}^{\omega} G(\sigma(s), s)p(s)\,\Delta s} < \lambda < \frac{1}{f_\infty \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s)p(s)\,\Delta s} \tag{14}$$

there exists at least one solution of (1) *and* (2) *in $\mathscr{P}$.*

Proof. Let λ be as in (14) and let $\eta > 0$ be such that

$$\frac{1}{(f_0 - \eta)kK \int_{\xi}^{\omega} G(\sigma(s), s)p(s)\,\Delta s} \leqslant \lambda \leqslant \frac{1}{(f_\infty + \eta) \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s)p(s)\,\Delta s}. \tag{15}$$

Let T be the completely continuous, cone-preserving operator defined in (12). We seek a fixed point of T in $\mathscr{P}$ by establishing the hypotheses of Theorem 2.

First, consider f_0. There exists an $H_1 > 0$ such that $f(x) \geqslant (f_0 - \eta)x$ for $0 < x \leqslant H_1$ by the definition of f_0. Pick $x \in \mathscr{P}$ with $\|x\| = H_1$. For $t \in [\xi, \sigma(\omega)]$, where ξ, ω is as in (7) and k as in (8), we have

$$x(t) \geqslant g(t)\|x\| = g(t)H_1 \geqslant kH_1. \tag{16}$$

Using the left-hand side of (15) and (16) and τ from (9), we get

$$
\begin{aligned}
(Tx)(\tau) &= \lambda \int_{t_1}^{\sigma(t_2)} G(\tau,s)p(s)f(x^\sigma(s))\,\Delta s \\
&\geqslant \lambda \int_{\xi}^{\omega} G(\tau,s)p(s)f(x^\sigma(s))\,\Delta s \\
&\geqslant \lambda(f_0-\eta)g(\tau)\int_{\xi}^{\omega} G(\sigma(s),s)p(s)x^\sigma(s)\,\Delta s \\
&\geqslant \lambda(f_0-\eta)H_1kK\int_{\xi}^{\omega} G(\sigma(s),s)p(s)\,\Delta s \\
&\geqslant H_1 \\
&= \|x\|.
\end{aligned}
$$

Therefore $\|Tx\| \geqslant \|x\|$. This prompts us to define

$$\Omega_1 := \{x \in \mathscr{B}\colon \|x\| < H_1\},$$

whereby our work above confirms

$$\|Tx\| \geqslant \|x\|, \quad x \in \mathscr{P} \cap \partial\Omega_1.$$

Next consider f_∞. Again by definition there exists an $\bar{H}_2 > H_1$ such that $f(x) \leqslant (f_\infty + \eta)x$ for $x \geqslant \bar{H}_2$. If f is bounded, there exists $M > 0$ with $f(x) \leqslant M$ for all $x \in (0,\infty)$. Let

$$H_2 := \max\left\{2\bar{H}_2, \lambda M \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)\,\Delta s\right\}.$$

If $x \in \mathscr{P}$ with $\|x\| = H_2$, then we have

$$
\begin{aligned}
(Tx)(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s \\
&\leqslant \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)f(x^\sigma(s))\,\Delta s \\
&\leqslant \lambda M \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)\,\Delta s \\
&\leqslant H_2 \\
&= \|x\|.
\end{aligned}
$$

As a result, $\|Tx\| \leqslant \|x\|$. Thus, take

$$\Omega_2 := \{x \in \mathscr{B}\colon \|x\| < H_2\}$$

so that $\|Tx\| \leqslant \|x\|$ for $x \in \mathscr{P} \cap \partial\Omega_2$. If f is unbounded, take $H_2 := \max\{2H_1, \bar{H}_2\}$ such that $f(x) \leqslant f(H_2)$ for $0 < x \leqslant H_2$. If $x \in \mathscr{P}$ with $\|x\| = H_2$, then we have

$$\begin{aligned}(Tx)(t) &= \lambda \int_{t_1}^{\sigma(t_2)} G(t,s)p(s)f(x^\sigma(s))\,\Delta s \\ &\leqslant \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)f(H_2)\,\Delta s \\ &\leqslant \lambda(f_\infty + \eta)H_2 \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s)p(s)\,\Delta s \\ &\leqslant H_2 \\ &= \|x\|,\end{aligned}$$

where the penultimate line follows from the left-hand side of (15). Hence, we have shown that

$$\|Tx\| \leqslant \|x\|, \quad x \in \mathscr{P} \cap \partial\Omega_2$$

if we take

$$\Omega_2 := \{x \in \mathscr{B} \colon \|x\| < H_2\}.$$

Once again an application of Theorem 2 validates the conclusion of the theorem. □

References

[1] R.P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, Results Math. 35 (1999) 3–22.

[2] R.P. Agarwal, M. Bohner, P. Wong, Sturm–Liouville eigenvalue problems on time scales, Appl. Math. Comput. 99 (1999) 153–166.

[3] R.P. Agarwal, M. Bohner, P. Wong, Eigenvalues and eigenfunctions of discrete conjugate boundary value problems, Comput. Math. Appl. 38 (1999) 159–183.

[4] D.R. Anderson, Discrete third-order three-point right focal boundary value problems, Comput. Math. Appl., to appear.

[5] D.R. Anderson, J.M. Davis, Multiple solutions and eigenvalues for third-order right-focal boundary value problems, J. Math. Anal. Appl., to appear.

[6] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, in: Nonlinear Dynamics and Quantum Dynamical Systems, Mathematical Research, Vol. 59, Akademie Verlag, Berlin, 1990.

[7] C.J. Chyan, J. Henderson, Eigenvalue problems for nonlinear differential equations on a measure chain, J. Math. Anal. Appl. 245 (2000) 547–559.

[8] C.J. Chyan, J.M. Davis, J. Henderson, W.K.C. Yin, Eigenvalue comparisons for nonlinear differential equations on a measure chain, Electron. J. Differential Equations 1998 (35) (1998) 1–7.

[9] J.M. Davis, J. Henderson, K.R. Prasad, W. Yin, Eigenvalue intervals for nonlinear right focal problems, Appl. Anal. 74 (2000) 215–231.

[10] K. Deimling, Nonlinear Functional Analysis, Springer, New York, 1985.

[11] P.W. Eloe, J. Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problems, Differential Equations Dynam. Systems 6 (1998) 309–317.

[12] L.H. Erbe, S. Hilger, Sturmian theory on measure chains, Differential Equations Dynam. Systems 1 (1993) 223–246.

[13] L.H. Erbe, A.C. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, Math. Comput. Modelling 32 (2000) 571–585.

[14] L.H. Erbe, A. Peterson, Green's functions and comparison theorems for differential equations on measure chains, Dynamics Contin. Discrete Impulsive Systems 6 (1999) 121–137.

[15] L.H. Erbe, A.C. Peterson, Eigenvalue conditions and positive solutions, J. Differ. Equations Appl. 6 (2000) 165–191.
[16] L.H. Erbe, S. Hu, H. Wang, Multiple positive solutions of some boundary value problems, J. Math. Anal. Appl. 184 (1994) 640–648.
[17] L.H. Erbe, M. Tang, Existence and multiplicity of positive solutions to nonlinear boundary value problems, Differential Equations Dynam. Systems 4 (1996) 313–320.
[18] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Resultate Math. 18 (1990) 18–56.
[19] J. Henderson, H. Wang, Positive solutions for nonlinear eigenvalue problems, J. Math. Anal. Appl. 208 (1997) 252–259.
[20] D. Jiang, H. Liu, Existence of positive solutions to $(k, n-k)$ conjugate boundary value problems, Kyushu J. Math. 53 (1998) 115–125.
[21] M.A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.