A $2n$TH ORDER LINEAR DIFFERENCE EQUATION

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ABSTRACT: We give a formulation of generalized zeros and (n, n)-disconjugacy for even order formally self-adjoint scalar linear difference equations. Sign conditions on the coefficient functions ensure (n, n)-disconjugacy. This and additional results are obtained with the use of an associated nonlinear operator.

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1 INTRODUCTION

In this paper we will be concerned with the $2n$th-order linear difference equation

$$\sum_{i=0}^{n} \Delta^i \left(r_i(t-i) \Delta^i y(t-i) \right) = 0 \tag{1}$$

for t in the discrete interval $[a, \infty) \equiv \{a, a+1, a+2, \ldots\}$, where $r_i(t)$ is real-valued on $[a+n-i, \infty)$ for $0 \le i \le n$, and

$$r_n(t) > 0 \tag{2}$$

on $[a, \infty)$. A function y defined on $[a, \infty)$ is a solution of (1), provided (1) holds for $t \ge a+n$. Note that Ahlbrandt and Peterson [1] studied (1) in the context of the discrete calculus of variations, and Peil and Peterson [2] studied the asymptotic behavior of solutions of (1) with $r_i(t) \equiv 0$ for $1 \le i \le n-1$. Here we will generalize some of the results in [2].

Definition For any function y defined on $[a, \infty)$, define for $t \ge a+n$ the nonlinear operator F by

$$\begin{aligned} Fy(t) \quad := \quad & (-1)^n \sum_{i=1}^{n} (-1)^i \Delta^{i-1} y(t-1) \left(r_i(t-1) \Delta^i y(t-1) \right) \\ & -(-1)^n \sum_{j=2}^{n} \sum_{i=j}^{n} (-1)^{i-j} \Delta^{i-j} y(t) \Delta^{j-1} \left(r_i(t-j) \Delta^i y(t-j) \right). \end{aligned} \tag{3}$$

Beginning in the proof of Lemma 2 we will extend the domain of the coefficient functions r_i to the integers $\mathbb{Z}$ by

$$r_i(t) \equiv r_i(a+n-i) \ \text{ when } \ t \le a+n-i \ \text{ for } \ 0 \le i \le n. \tag{4}$$

Then for any function y defined on $\mathbb{Z}$, $Fy(t)$ is defined for all $t \in \mathbb{Z}$ by (3) as well.

Lemma 1 *If y is a solution of* (1)*, then*

$$\Delta Fy(t) = (-1)^n r_0(t)[y(t)]^2 + \sum_{i=1}^{n}(-1)^{n+i} r_i(t-1)\left[\Delta^i y(t-1)\right]^2. \tag{5}$$

In particular, if

$$(-1)^{n+i} r_i(t) \geq 0 \quad for \quad t \geq a+n-i, \;\; 0 \leq i \leq n-1, \tag{6}$$

then F is nondecreasing along solutions y of (1) *for $t \geq a+n$.*

Proof: Assume y is a solution of (1). Then

$$\begin{aligned}
\Delta Fy(t) \;=\; & (-1)^n \sum_{i=1}^{n}(-1)^i \Delta^{i-1} y(t) \Delta \Big(r_i(t-1)\Delta^i y(t-1)\Big) && (7)\\
& +(-1)^n \sum_{i=1}^{n}(-1)^i \Delta^i y(t-1) \Big(r_i(t-1)\Delta^i y(t-1)\Big) && (8)\\
& -(-1)^n \sum_{j=2}^{n}\sum_{i=j}^{n}(-1)^{i-j}\Delta^{i-j} y(t)\Delta^j \Big(r_i(t-j)\Delta^i y(t-j)\Big) && (9)\\
& -\sum_{j=2}^{n}\sum_{i=j}^{n}(-1)^{n+i-j}\Delta^{i-j+1} y(t)\Delta^{j-1}\Big(r_i(t-j+1)\Delta^i y(t-j+1)\Big), && (10)
\end{aligned}$$

where (7) and (8) result from the product rule of the first advanced times the difference of the second plus the second times the difference of the first, while (9) and (10) come from the product rule of the first times the difference of the second plus the difference of the first times the second advanced. Next break off the $j=2$ terms to rewrite (10) as

$$\begin{aligned}
& -(-1)^n \sum_{j=3}^{n}\sum_{i=j}^{n}(-1)^{i-j}\Delta^{i-j+1} y(t)\Delta^{j-1}\Big(r_i(t-j+1)\Delta^i y(t-j+1)\Big) && (11)\\
& -(-1)^n \sum_{i=2}^{n}(-1)^{i-2}\Delta^{i-1} y(t)\Delta\Big(r_i(t-1)\Delta^i y(t-1)\Big). && (12)
\end{aligned}$$

Since (12) subtracts off all terms in (7) save when $i=1$, we have

$$\begin{aligned}
\Delta Fy(t) = & (-1)^{n+1} y(t)\Delta\left(r_1(t-1)\Delta y(t-1)\right) && (13)\\
& +\sum_{i=1}^{n}(-1)^{n+i} r_i(t-1)\left[\Delta^i y(t-1)\right]^2 && (14)\\
& -(-1)^n \sum_{j=2}^{n}\sum_{i=j}^{n}(-1)^{i-j}\Delta^{i-j} y(t)\Delta^j\Big(r_i(t-j)\Delta^i y(t-j)\Big) && (15)\\
& -(-1)^n \sum_{j=3}^{n}\sum_{i=j}^{n}(-1)^{i-j}\Delta^{i-j+1} y(t)\Delta^{j-1}\Big(r_i(t-j+1)\Delta^i y(t-j+1)\Big). && (16)
\end{aligned}$$

From (15) we isolate the $i=j$ terms, so that (15) becomes

$$\begin{aligned}
& -(-1)^n \sum_{j=2}^{n-1}\sum_{i=j+1}^{n}(-1)^{i-j}\Delta^{i-j} y(t)\Delta^j\Big(r_i(t-j)\Delta^i y(t-j)\Big) && (17)\\
& -(-1)^n \sum_{i=2}^{n} y(t)\Delta^i\Big(r_i(t-i)\Delta^i y(t-i)\Big). && (18)
\end{aligned}$$

Note that (16) cancels (17) by replacing $j-1$ by j in (16). Consequently, using lines (13), (14), and (18), we have

$$\Delta Fy(t) = (-1)^{n+1} y(t) \sum_{i=1}^{n} \Delta^i \Big(r_i(t-i) \Delta^i y(t-i) \Big) + \sum_{i=1}^{n} (-1)^{n+i} r_i(t-1) \left[\Delta^i y(t-1) \right]^2 .$$

Since y is a solution of (1),

$$\Delta Fy(t) = (-1)^n y(t) \cdot r_0(t) y(t) + \sum_{i=1}^{n} (-1)^{n+i} r_i(t-1) \left[\Delta^i y(t-1) \right]^2 ,$$

so that (5) follows. Moreover, if (6) holds, then $\Delta Fy(t) \geq 0$ on $[a+n, \infty)$. Therefore F is nondecreasing along solutions of (1) for $t \geq a+n$.

□

Definitions As in [2], y is a *type I solution* of (1) if y solves (1) and $Fy(t) \leq 0$ in a neighborhood of ∞. If $Fy(t) > 0$ near ∞, then y is a *type II solution.* In view of Lemma 1, all solutions of (1) are type I or type II solutions if (6) holds.

Following [1], we say that a solution y of (1) has a *generalized zero of order (at least) k at a* provided $y(a+m) = 0$ for $0 \leq m \leq k-1$, and a *generalized zero of order (at least) k at $t_0 > a$* provided

$$y(t_0 - 1) \neq 0, \qquad y(t_0 + m) = 0 \qquad \text{for} \qquad 0 \leq m \leq k-2,$$

and

$$(-1)^k y(t_0 - 1) y(t_0 + k - 1) \geq 0.$$

Equation (1) is *(n,n)-disconjugate* on $[a, \infty)$ provided there is no nontrivial solution of (1) with two or more generalized zeros of order (at least) n on $[a, \infty)$.

Lemma 2 *Suppose* (6) *holds, and that y is a nontrivial solution of* (1) *with a generalized zero of order (at least) n at $t_0 \geq a$.*

(*a*) *If $t_0 > a$, then*

$$Fy(t_0) < 0, \tag{19}$$

and either

(*i*) $Fy(t_0 + 1) > 0$, *if* $(-1)^n y(t_0 - 1) y(t_0 + n - 1) > 0$,

or

(*ii*) *y has exactly $n+k$ consecutive zeros starting at t_0, where $0 \leq k \leq n-1$, and*

$$Fy(t_0 + m) = 0, \;\; 1 \leq m \leq k+1, \tag{20}$$

with

$$Fy(t_0 + k + 2) > 0. \tag{21}$$

(*b*) *If $t_0 = a$, then case* (*a*) (*ii*) *holds, and* (20), (21) *both follow.*

Proof: Assume y is a nontrivial solution of (1). For part (a), assume that y has a generalized zero of order (at least) n at $t_0 > a$. Then, by definition,

$$y(t_0+m)=0 \;\text{ for }\; 0\le m\le n-2 \tag{22}$$

and

$$y(t_0-1)\neq 0, \;\text{ while }\; (-1)^n y(t_0-1)y(t_0+n-1)\ge 0. \tag{23}$$

As a result of (22),

$$\Delta^i y(t_0)=0 \;\text{ for }\; 0\le i\le n-2. \tag{24}$$

To ensure that everything is defined on the same intervals, extend the domains of the coefficient functions r_i to the integers as in (4). It suffices to show that (1) satisfies the lemma—with these new coefficients—on the integers. Note that $Fy(t)$ is now defined and nondecreasing along solutions y of (1) on the integers as well. Now, using (3), (22), and (24), we see that

$$\begin{aligned}
Fy(t_0) &= (-1)^n\sum_{i=1}^{n}(-1)^i\Delta^{i-1}y(t_0-1)\Big(r_i(t_0-1)\Delta^i y(t_0-1)\Big)\\
&\quad +(-1)^{n+1}\sum_{j=2}^{n}\sum_{i=j}^{n}(-1)^{i-j}\Delta^{i-j}y(t_0)\Delta^{j-1}\Big(r_i(t_0-j)\Delta^i y(t_0-j)\Big)\\
&= (-1)^n\sum_{i=1}^{n}(-1)^i\Delta^{i-1}y(t_0-1)\Big(r_i(t_0-1)\Delta^i y(t_0-1)\Big)\\
&= \sum_{i=1}^{n-1}(-1)^{n+i}(-1)^{i-1}y(t_0-1)\Big(r_i(t_0-1)\cdot(-1)^i y(t_0-1)\Big)\\
&\quad +(-1)^{n-1}y(t_0-1)r_n(t_0-1)\left[y(t_0+n-1)+(-1)^n y(t_0-1)\right]\\
&= \sum_{i=1}^{n-1}(-1)^{n+i-1}r_i(t_0-1)\left[y(t_0-1)\right]^2-r_n(t_0-1)\left[y(t_0-1)\right]^2\\
&\quad +r_n(t_0-1)\cdot(-1)^{n-1}y(t_0-1)y(t_0+n-1)\\
&= -\left[y(t_0-1)\right]^2\sum_{i=1}^{n}(-1)^{n+i}r_i(t_0-1)\\
&\quad -r_n(t_0-1)(-1)^n y(t_0-1)y(t_0+n-1).
\end{aligned} \tag{25}$$

By (2), (6), and (23), equation (25) yields the inequality (19).

To show case $(a)(i)$, suppose we have (22) and

$$(-1)^n y(t_0-1)y(t_0+n-1)>0. \tag{26}$$

Then

$$\begin{aligned}
Fy(t_0+1) &= (-1)^n\sum_{i=1}^{n}(-1)^i\Delta^{i-1}y(t_0)\Big(r_i(t_0)\Delta^i y(t_0)\Big)\\
&\quad -\sum_{j=2}^{n}\sum_{i=j}^{n}(-1)^{n+i-j}\Delta^{i-j}y(t_0+1)\Delta^{j-1}\Big(r_i(t_0+1-j)\Delta^i y(t_0+1-j)\Big).
\end{aligned} \tag{27}$$

From (22) we know that

$$\Delta^{i-1}y(t_0)=0 \;\text{ for }\; 1\le i\le n-1,$$

and $\Delta^{i-j}y(t_0+1)=0$ except when $i=n$ and $j=2$. As a result, (27) becomes

$$\begin{aligned}
Fy(t_0+1) &= (-1)^n(-1)^n\Delta^{n-1}y(t_0)\left(r_n(t_0)\Delta^n y(t_0)\right)\\
&\quad -(-1)^n(-1)^{n-2}\Delta^{n-2}y(t_0+1)\Delta\left(r_n(t_0-1)\Delta^n y(t_0-1)\right)\\
&= y(t_0+n-1)r_n(t_0)\Delta^n y(t_0)\\
&\quad -y(t_0+n-1)\Delta\left(r_n(t_0-1)\Delta^n y(t_0-1)\right)\\
&= y(t_0+n-1)r_n(t_0-1)\Delta^n y(t_0-1)\\
&= y(t_0+n-1)r_n(t_0-1)\left[y(t_0+n-1)+(-1)^n y(t_0-1)\right]\\
&= r_n(t_0-1)\left(y(t_0+n-1)\right)^2\\
&\quad +r_n(t_0-1)\cdot(-1)^n y(t_0-1)y(t_0+n-1)\\
&> 0
\end{aligned}$$

by (2) and (26).

For case $(a)(ii)$, we assume there is a $k\geq 0$ such that

$$y(t_0+\tau)=0 \quad\text{for}\quad 0\leq\tau\leq n+k-1; \tag{28}$$

since y is a nontrivial solution, $k\leq n-1$. Then, for $m\in\{1,\ldots,k+1\}$, (28) implies that

$$\begin{aligned}
Fy(t_0+m) &= (-1)^n\sum_{i=1}^{n}(-1)^i\Delta^{i-1}y(t_0+m-1)\Big(r_i(t_0+m-1)\Delta^i y(t_0+m-1)\Big)\\
&\quad -(-1)^n\sum_{j=2}^{n}\sum_{i=j}^{n}(-1)^{i-j}\Delta^{i-j}y(t_0+m)\Delta^{j-1}\Big(r_i(t_0+m-j)\Delta^i y(t_0+m-j)\Big)\\
&= 0.
\end{aligned}$$

However,

$$\begin{aligned}
Fy(t_0+k+2) &= (-1)^n\sum_{i=1}^{n}(-1)^i\Delta^{i-1}y(t_0+k+1)\Big(r_i(t_0+k+1)\Delta^i y(t_0+k+1)\Big)\\
&\quad -(-1)^n\sum_{j=2}^{n}\sum_{i=j}^{n}\Big\{(-1)^{i-j}\Delta^{i-j}y(t_0+k+2)\\
&\quad \cdot\Delta^{j-1}\Big(r_i(t_0+k+2-j)\Delta^i y(t_0+k+2-j)\Big)\Big\}\\
&= (-1)^n(-1)^n\Delta^{n-1}y(t_0+k+1)\left(r_n(t_0+k+1)\Delta^n y(t_0+k+1)\right)\\
&\quad -(-1)^n(-1)^{n-2}\Delta^{n-2}y(t_0+k+2)\Delta\left(r_n(t_0+k)\Delta^n y(t_0+k)\right)\\
&= y(t_0+n+k)r_n(t_0+k+1)\Delta^n y(t_0+k+1)\\
&\quad -y(t_0+n+k)\Delta\left(r_n(t_0+k)\Delta^n y(t_0+k)\right)\\
&= y(t_0+n+k)r_n(t_0+k)\Delta^n y(t_0+k)\\
&= r_n(t_0+k)\left(y(t_0+n+k)\right)^2\\
&> 0
\end{aligned}$$

using (2) and (28).

Finally, in case (b), the fact that y has a generalized zero of order (at least) n at a implies that case $(a)(ii)$ holds with $t_0=a$. □

2 MAIN RESULTS

The following result, which is important in the discrete calculus of variations, was first proven by Ahlbrandt and Peterson [1]. Here it follows easily from Lemma 2.

Theorem 3 *Assume* (6) *holds. Then* (1) *is* (n, n)*-disconjugate on* $[a, \infty)$.

Proof: Suppose y is a nontrivial solution with a generalized zero of order exactly n at $t_1 > a$. Then, by Lemma 2, either $Fy(t_1+1) > 0$ or $Fy(t_1+2) > 0$, so that $Fy(t) > 0$ on $[t_1 + 2, \infty)$. If y had another generalized zero of order exactly n at $t_2 \geq t_1 + n$, then (19) would imply that $Fy(t_2) < 0$, a contradiction because $t_2 \in [t_1 + 2, \infty)$. A similar contradiction is reached if y has a generalized zero of order greater than n at t_1 or t_2 or both. Hence, a nontrivial solution of (1) can have at most one generalized zero of order (at least) n on $[a, \infty)$, so that (1) is (n, n)-disconjugate on $[a, \infty)$. □

Theorem 4 *Assume* (6) *holds. Then any nontrivial solution of* (1) *with a generalized zero of order (at least)* n *is a type II solution. In particular,* (1) *has* $2n$ *linearly-independent type II solutions.*

Proof: Assume y is a nontrivial solution of (1) with a generalized zero of order (at least) n at t_0 for some $t_0 \geq a$. Again, extend the domains of the coefficient functions r_i as in (4), and note that $Fy(t)$ is defined and nondecreasing along solutions on the integers, due to Lemma 1. Then, using Lemma 2, we see that either $Fy(t_0 + 1) > 0$, or, if y has $n+k$ zeros starting at t_0 for some $0 \leq k \leq n-1$, then $Fy(t_0+k+2) > 0$. Either way, there exists $t_1 > t_0$ such that $Fy(t_1) > 0$, and so $Fy(t) > 0$ on $[t_1, \infty)$. Hence, y is a type II solution.

We now show that there are $2n$ linearly-independent type II solutions of (1). Let y_k, $0 \leq k \leq 2n-1$, be solutions of (1) satisfying

$$y_k(a+i) = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k, \quad 0 \leq i \leq 2n-1. \end{cases}$$

Because the y_k are nontrivial solutions of (1) with at least n consecutive zeros on $[a, a+2n-1]$, we have by the above part of the proof that the y_k are type II solutions for $0 \leq k \leq 2n-1$. It is clear that these solutions are linearly independent. □

Theorem 5 *If* (6) *holds, then* (1) *has* n *linearly-independent type I solutions.*

Proof: Following the proof of Theorem 4 in [2], for each fixed $s \geq a+n$, we let $v_k(t, s)$ be a nontrivial solution of (1) for $1 \leq k \leq n$, satisfying the $2n-1$ boundary conditions

$$\begin{aligned} v_k(a+i, s) &= 0 \quad \text{for } 0 \leq i \leq n-1 \text{ but } i \neq k-1 \\ v_k(s+i, s) &= 0 \quad \text{for } 0 \leq i \leq n-1. \end{aligned} \tag{29}$$

Then define

$$u_k(t, s) := \frac{v_k(t, s)}{\sqrt{v_k^2(a, s) + v_k^2(a+1, s) + \cdots + v_k^2(a+2n-1, s)}} \tag{30}$$

for $1 \le k \le n$ and $s \ge a+n$. Then $u_k(t,s)$ is a solution of (1) satisfying

$$\sum_{i=0}^{2n-1} u_k^2(a+i,s) = 1.$$

Thus, for each k, the sequence

$$\{u_k(a,s), u_k(a+1,s), \dots, u_k(a+2n-1,s)\}_{s=a+n}^{\infty}$$

has a convergent subsequence

$$\{u_k(a,s_{j_k}), u_k(a+1,s_{j_k}), \dots, u_k(a+2n-1,s_{j_k})\}_{j=1}^{\infty}.$$

Let

$$v_{i+1,k} := \lim_{j\to\infty} u_k(a+i,s_{j_k}) \tag{31}$$

for $0 \le i \le 2n-1$. Then

$$\sum_{i=0}^{2n-1} v_{i+1,k}^2 = 1. \tag{32}$$

Further, let y_k be the solutions of (1) satisfying

$$y_k(a+i) = v_{i+1,k} \tag{33}$$

for $0 \le i \le 2n-1$, $1 \le k \le n$; note that the y_k are nontrivial solutions by (32) and (33). Since $v_k(t,s)$ has n consecutive zeros starting at s by (29), formula (30) implies the same for $u_k(t,s)$, $1 \le k \le n$. Hence, by case $(a)(ii)$ of Lemma 2, we have

$$Fu_k(s_{j_k}+1, s_{j_k}) = 0;$$

as $Fu_k(t,s_{j_k})$ is nondecreasing,

$$Fu_k(t,s_{j_k}) \le 0$$

on $[a+n, s_{j_k}+1]$ for $1 \le k \le n$. So, for each $t_1 \ge a+n$, there exists j_{t_1} such that $s_{j_k} > t_1$ for all $j_k \ge j_{t_1}$. Then

$$Fu_k(t_1,s_{j_k}) \le 0$$

for all $j_k \ge j_{t_1}$. Taking the limit as $j \to \infty$, we get that

$$Fy_k(t_1) \le 0$$

for $1 \le k \le n$. As $t_1 \ge a+n$ was arbitrary,

$$Fy_k(t) \le 0$$

for all $t \ge a+n$, for $1 \le k \le n$. It follows that the y_k are type I solutions of (1) for $1 \le k \le n$. By (29),(30),(31), and (33) we have that

$$y_k(a+i) = 0 \ \text{ for } \ 0 \le i \le n-1 \ \text{ if } \ i \ne k-1. \tag{34}$$

If $y_k(a+k-1) = 0$, then y_k would have n consecutive zeros and thus a generalized zero of order (at least) n at a, so that y_k would be a type II solution by Theorem 4. Consequently, $y_k(a+k-1) \ne 0$ for $1 \le k \le n$, and the y_k are linearly independent. $\square$

Theorem 6 *If* (6) *holds and* y *is a type I solution of* (1), *then*

$$\sum_{t=a+n-i}^{\infty} (-1)^{n-i} r_i(t) \left[\Delta^i y(t)\right]^2 < \infty, \tag{35}$$

for $0 \leq i \leq n$. *If* $r_0(t) \neq 0$ *in a neighborhood of* ∞, *then every nontrivial type I solution of* (1) *is a strict type I solution.*

Proof: Let y be a type I solution of (1); then $Fy(t) \leq 0$ for $t \geq a+n$. Let

$$M := \lim_{t\to\infty} Fy(t) \leq 0.$$

Summing both sides of (5) from $a+n$ to ∞ yields

$$M - Fy(a+n) = \sum_{t=a+n}^{\infty} \left((-1)^n r_0(t)[y(t)]^2 + \sum_{i=1}^{n} (-1)^{n+i} r_i(t-1) \left[\Delta^i y(t-1)\right]^2 \right).$$

Thus (35) holds. Now assume $r_0(t) \neq 0$ in a neighborhood of ∞, and that v is a nontrivial type I solution of (1); then $Fv(t) \leq 0$ for $t \geq a+n$. Suppose there exists a $t_0 \geq a+n$ such that $Fv(t_0) = 0$. Then $Fv(t) \equiv 0$ on $[t_0, \infty)$, by Lemma 1. But then $\Delta Fv(t) = 0$ on $[t_0, \infty)$, so that from (5) we have

$$(-1)^n r_0(t)[v(t)]^2 + \sum_{i=1}^{n} (-1)^{n+i} r_i(t-1) \left[\Delta^i v(t-1)\right]^2 \equiv 0$$

for $t \geq t_0$. Since (6) holds all terms are nonnegative; moreover, $r_0(t) \neq 0$ near ∞ gives that v is the trivial solution, contrary to assumption. Therefore, $Fv(t) < 0$ for all $t \geq a+n$, and v is a strict type I solution of (1). □

Corollary 7 *If* (6) *holds and* $\liminf_{t\to\infty} (-1)^n r_0(t) > 0$, *then* (1) *has* n *linearly-independent type I solutions* v_k *satisfying* $\lim_{t\to\infty} v_k(t) = 0$ *for* $1 \leq k \leq n$.

Proof: By Theorem 5, equation (1) has n linearly-independent type I solutions $v_1, \ldots, v_n$. For any $k \in \{1, \ldots, n\}$, (35) implies that

$$(-1)^n \sum_{t=a+n}^{\infty} r_0(t)[v_k(t)]^2 < \infty,$$

so that

$$\lim_{t\to\infty} r_0(t)[v_k(t)]^2 = 0.$$

Since $\liminf_{t\to\infty} (-1)^n r_0(t) > 0$, we have that $\lim_{t\to\infty} v_k(t) = 0$, for $1 \leq k \leq n$. □

References

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