

# Green's Function for an $n$ -point Right Focal Boundary Value Problem

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## Abstract

We determine sufficient conditions for the positivity of the Green's function for a certain  $n$ -point right focal boundary value problem. For  $n = 3, 4, 5$  we see that positivity depends on the distances between boundary points.

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## 1 Preliminaries

In this paper we will be concerned with the operator  $L$  defined by

$$Ly(t) := (-1)^{n-1} \Delta^n y(t-k), \quad (1)$$

where  $t$ ,  $n$ , and  $k$  are from  $\mathbb{Z}$  (the set of integers), with  $n \geq 2$  and  $k \geq 1$ . Then the  $n$ -point right focal boundary value problem that we will consider is given by

$$Ly(t) = 0, \quad \text{for } t \in \mathbb{Z}, \quad (2)$$

with

$$\Delta^{r-1} y(t_r) = 0, \quad \text{where } a = t_1 < t_2 < \cdots < t_{n-1} < t_n = b + 1. \quad (3)$$

We note that solutions to (2) are defined on  $\mathbb{Z}$ . The following lemma, however, concerns solutions  $y$  to the corresponding nonhomogeneous  $n$ -point right focal boundary value problem

$$Ly(t) = h(t)$$

for  $t \in [a+k, b+k]$  with boundary conditions as in (3), where  $h$  is some real-valued function defined on  $[a+k, b+k]$  for  $k \in \{1, 2, \dots, n\}$ . The proof is from Diaz [2].

**Lemma 1** *Suppose a function  $G_n(t, s)$  has the properties:*

- (i)  $G_n(t, s)$  is defined on  $[a, b+n] \times [a+k, b+k]$ ;
- (ii) For each fixed  $s \in [a+k, b+k]$ ,  $LG_n(t, s) = \delta_{ts}$  for all  $t \in [a, b+n]$ , where  $\delta_{ts}$  is the Kronecker delta function;
- (iii) For each fixed  $s \in [a+k, b+k]$ ,  $\Delta^{r-1} G_n(t_r, s) = 0$ ,  $r = 1, 2, \dots, n$ .

Then, for  $h(t)$  defined on  $[a + k, b + k]$ , where  $k \in \{1, 2, \dots, n\}$ , we have that

$$y(t) = \sum_{s=a+k}^{b+k} G_n(t, s)h(s)$$

solves  $Ly(t) = h(t)$  with the boundary conditions given in (3).

*Proof:* For  $t \in [a + k, b + k]$ ,

$$\begin{aligned} Ly(t) &= L \left( \sum_{s=a+k}^{b+k} G_n(t, s)h(s) \right) \\ &= \sum_{s=a+k}^{b+k} LG_n(t, s)h(s) \\ &= \sum_{s=a+k}^{b+k} \delta_{ts}h(s) \\ &= h(t). \end{aligned}$$

The boundary conditions are satisfied by condition (iii) in the statement of the lemma.  $\square$

The *Green's function* for the  $n$ -point right focal boundary value problem given by (2) and (3) is defined to be the unique function satisfying properties (i) through (iii) given in Lemma 1. The development of this Green's function will be given in the next two sections, and summarized in Lemma 2.

## 2 The Factorial Function

We will now need what is known as the *factorial function*. This function,  $t^{(j)}$ , is defined as follows:

- (i) if  $j = 1, 2, 3, \dots$ , then  $t^{(j)} = t(t-1)(t-2) \cdots (t-j+1)$ ;
- (ii) if  $j = 0$ , then  $t^{(0)} = 1$ ;
- (iii) if  $j = -1, -2, -3, \dots$ , then  $t^{(j)} = \frac{1}{(t+1)(t+2) \cdots (t-j)}$ ;
- (iv) if  $j$  is not an integer, then  $t^{(j)} = \frac{\Gamma(t+1)}{\Gamma(t-j+1)}$ , where  $\Gamma(t)$  is the gamma function.

We note from (i) that for  $j$  a positive integer, we have the well-known formula

$$\Delta t^{(j)} = j t^{(j-1)}.$$

When  $s \in [t_1 + k, t_n + k - 1] = [a + k, b + k]$  for some integer  $k \geq 1$ , define

$$y_j(t, s) := \frac{1}{j!} (t - s)^{(j)} \tag{4}$$

for  $j = 1, 2, \dots, n - 1$ ; by convention  $y_0(t, s) \equiv 1$  for all  $t$  and  $s$ . Then

$$\Delta_t^i y_j(t, s) = y_{j-i}(t, s) \quad (5)$$

for  $i \leq j$ , and

$$\Delta_t^i y_j(t, s) \equiv 0$$

for  $i > j$ . Thus,  $y_j(t, s)$  is a solution to the initial value problem

$$Ly_j(t, s) = 0, \quad \Delta^i y_j(s, s) = \delta_{ij},$$

for  $j = 1, 2, \dots, n-1$  and  $i = 0, 1, \dots, n-1$ . As a result, a set of  $n$  linearly-independent solutions to  $Ly = 0$  for  $L$  as in (1) is  $\{1, y_1(t, s), \dots, y_{n-1}(t, s)\}$ . Moreover, for fixed  $t$ ,  $y_j(t, s)$  can be thought of as a function of  $s$ ; it is easy to verify that

$$y_j(t, s) = (-1)^j y_j(s + j - 1, t). \quad (6)$$

### 3 Green's Function for the Right Focal Problem

Following Hartman [3], a solution  $y$  of  $Ly(t) = 0$  has a *generalized zero at a* provided  $y(a) = 0$ , and a *generalized zero at  $t_0 > a$*  if either  $y(t_0) = 0$  or there is an integer  $k$ ,  $1 \leq k \leq t_0 - a$ , such that  $(-1)^k y(t_0 - k)y(t_0) > 0$  and, if  $k > 1$ ,  $y(t_0 - k + 1) = \dots = y(t_0 - 1) = 0$ . The difference equation  $Ly(t) = 0$  is said to be *right disfocal* on the interval  $[a, b + n]$  if there does not exist a nontrivial solution  $y$  of  $Ly(t) = 0$  and points  $t_1 \leq t_2 \leq \dots \leq t_n \in [a, b + 1]$  such that  $\Delta^{i-1}y$  has a generalized zero at  $t_i$ ,  $1 \leq i \leq n$ . Note that when  $L$  is defined as in (1),  $Ly(t) = 0$  is right disfocal over any interval.

For integers  $n \geq 2$  and  $k \geq 1$ , and for  $i = 1, 2, \dots, n - 1$ , define

$$u_{n,i}(t, s) \equiv u_{n,i}(t, s : t_1, t_2, \dots, t_n),$$

with  $t, s$ , and  $t_j$  all integers for  $1 \leq j \leq n$ , as follows:

$$u_{n,i}(t, s) := (-1)^{n+1} \begin{vmatrix} 0 & y_1(t, t_1) & y_2(t, t_1) & \cdots & y_{n-1}(t, t_1) \\ h_2(s, i) & 1 & y_1(t_2, t_1) & \cdots & y_{n-2}(t_2, t_1) \\ h_3(s, i) & 0 & 1 & \cdots & y_{n-3}(t_3, t_1) \\ h_4(s, i) & 0 & 0 & \cdots & y_{n-4}(t_4, t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}, \quad (7)$$

where

$$h_j(s, i) := H(j - 1 - i)y_{n-j}(t_j, s - k + 1) \quad (8)$$

for  $j = 2, 3, \dots, n - 1$  and  $i = 1, 2, \dots, n - 1$ . Here

$$H(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x \geq 0 \end{cases}$$

is the usual Heaviside function, and  $y_j(t, s)$  is as defined in (4). In addition, define

$$v_{n,i}(t, s) := u_{n,i}(t, s) + (-1)^{n-1}y_{n-1}(t, s - k + 1) \quad (9)$$

for integers  $n \geq 2$ ,  $k \geq 1$ , and for  $i = 1, 2, \dots, n - 1$ .

Note that  $u_{n,i}(t, s)$  and  $v_{n,i}(t, s)$  solve (2) for all integers  $t$ . In the following lemma we will show that  $u_{n,i}$  and  $v_{n,i}$  compose the constituent parts of the Green's function for the  $n$ -point right focal boundary value problem given by (2) and (3). Ordinarily we would consider the Green's function  $G_n(t, s)$  for  $(t, s)$  in  $[a, b + n] \times [a + k, b + k]$ , as in Lemma 1. In the subsequent development, however, we will need to consider  $G_n(t, s)$  on an extended domain, allowed by the existence of  $u_{n,i}(t, s)$  and  $v_{n,i}(t, s)$  for all  $t \in \mathbb{Z}$ . The proof, due to Diaz [2] on the traditional domain, is thus established for the necessary extended domain:

**Lemma 2** For  $u_{n,i}(t, s) = u_{n,i}(t, s : t_1, t_2, \dots, t_n)$  as in (7) and  $v_{n,i}(t, s) = v_{n,i}(t, s : t_1, t_2, \dots, t_n)$  as in (9),

$$G_n(t, s : t_1, \dots, t_n) = \begin{cases} s \in I_1 & : \begin{cases} u_{n,1}(t, s) & : t < s - k + n \\ v_{n,1}(t, s) & : t \geq s - k + 1 \end{cases} \\ s \in I_2 & : \begin{cases} u_{n,2}(t, s) & : t < s - k + n \\ v_{n,2}(t, s) & : t \geq s - k + 1 \end{cases} \\ \vdots & \vdots \\ s \in I_{n-1} & : \begin{cases} u_{n,n-1}(t, s) & : t < s - k + n \\ v_{n,n-1}(t, s) & : t \geq s - k + 1, \end{cases} \end{cases} \quad (10)$$

where  $I_1 = [t_1 + k, t_2 + k - 1]$  and  $I_i = [t_i + k - 1, t_{i+1} + k - 1]$  for  $i = 1, 2, \dots, n - 1$ , satisfies properties (i) through (iii) in Lemma 1, and hence is the Green's function for the  $n$ -point right focal boundary value problem given in (2) and (3).

*Proof:* We will actually show that  $G_n(t, s)$  as given in (10) is defined on  $\mathbb{Z} \times [a + k, b + k]$ . First, note that from (7) and (9) we have

$$v_{n,i}(t, s) - u_{n,i}(t, s) = (-1)^{n-1}y_{n-1}(t, s - k + 1).$$

By (4),

$$y_{n-1}(s - k + j, s - k + 1) = 0$$

for  $j = 1, 2, \dots, n - 1$ ; thus,  $v_{n,i}(t, s) = u_{n,i}(t, s)$  for  $t \in [s - k + 1, s - k + n - 1]$ ,  $s \in I_i$ , for  $i = 1, 2, \dots, n - 1$ . Since  $a = t_1$  and  $t_n = b + 1$ ,

$$s \in [t_1 + k, t_n + k - 1] = [a + k, b + k],$$

and (i) of Lemma 1 is satisfied, here with  $t \in \mathbb{Z}$ .

Now let  $s \in [a + k, b + k]$ ; then  $s \in I_i$  for some  $i \in \{1, 2, \dots, n - 1\}$ . For  $t < s$ , we have  $t - k < s - k < s - k + n$ , and so from (10) and (1),

$$\begin{aligned} LG_n(t, s) &= (-1)^{n-1}\Delta_t^n G_n(t - k, s) \\ &= (-1)^{n-1}\Delta_t^n u_{n,i}(t - k, s) \\ &= Lu_{n,i}(t, s) \\ &= 0, \end{aligned}$$

since  $u_{n,i}(t, s)$  is a solution of  $Ly(t) = 0$ . If  $t > s$ , then  $t - k > s - k$ ; in other words,  $t - k \geq s - k + 1$ . Again by (10),

$$\begin{aligned} LG_n(t, s) &= (-1)^{n-1} \Delta_t^n G_n(t - k, s) \\ &= (-1)^{n-1} \Delta_t^n v_{n,i}(t - k, s) \\ &= Lv_{n,i}(t, s) \\ &= 0, \end{aligned}$$

likewise since  $v_{n,i}(t, s)$  is a solution of  $Ly(t) = 0$ . Suppose finally that  $t = s \in I_i$ , for some  $i \in \{1, 2, \dots, n-1\}$ . Because  $s - k + j < s - k + n$  for  $j \in \{0, 1, \dots, n-1\}$ , we have from (10) that

$$G_n(s - k + j, s) = u_{n,i}(s - k + j, s)$$

for  $j \in \{0, 1, \dots, n-1\}$ , and

$$G_n(s - k + n, s) = v_{n,i}(s - k + n, s).$$

Note that for any function  $y$  we have

$$\Delta^n y(t) = \sum_{j=0}^n (-1)^j \binom{n}{j} y(t + n - j) \quad (11)$$

by the definition of the forward difference. Then, using (11), we get

$$\begin{aligned} LG_n(s, s) &= (-1)^{n-1} \Delta^n G_n(s - k, s) \\ &= (-1)^{n-1} \sum_{j=0}^n (-1)^j \binom{n}{j} G_n(s - k + n - j, s) \\ &= (-1)^{n-1} v_{n,i}(s - k + n, s) \\ &\quad + (-1)^{n-1} \sum_{j=1}^n (-1)^j \binom{n}{j} u_{n,i}(s - k + n - j, s) \\ &= (-1)^{n-1} [v_{n,i}(s - k + n, s) - u_{n,i}(s - k + n, s)] \\ &\quad + (-1)^{n-1} \sum_{j=0}^n (-1)^j \binom{n}{j} u_{n,i}(s - k + n - j, s) \\ &= (-1)^{n-1} [(-1)^{n-1} y_{n-1}(s - k + n, s - k + 1)] \\ &\quad + (-1)^{n-1} \Delta^n u_{n,i}(s - k, s) \\ &= 1 + 0 \\ &= 1. \end{aligned}$$

Since  $s$  was an arbitrary element of  $[t_1 + k, t_n + k - 1]$ , we have that  $LG_n(t, s) = \delta_{ts}$ , which is (ii) of Lemma 1, proven here for all  $t \in \mathbb{Z}$ .

Lastly, we need to show that for each fixed  $s \in [t_1 + k, t_n + k - 1]$ ,

$$\Delta^{r-1} G_n(t_r, s) = 0$$

for  $r = 1, 2, \dots, n$ . Fix  $s \in I_i$  for some  $i \in \{1, 2, \dots, n-1\}$ . Since  $t_1 = a$  we have  $t_1 + k \leq s$ , so that  $t_1 \leq s - k < s - k + n$ . Hence from (10) we see that

$$G_n(t_1, s) = u_{n,i}(t_1, s).$$

But  $y_j(t_1, t_1) = 0$  for  $j = 1, 2, \dots, n-1$ , and so the top row of the determinant in (7) is all zeros; thus,

$$G_n(t_1, s) = 0.$$

For  $r = 2, 3, \dots, i$ , consider  $\Delta^{r-1}G_n(t_r, s)$  with  $t_r \leq t_i$ . Then  $2 \leq r \leq i$ , and

$$\begin{aligned} t_r &\leq t_i + i - 1 \\ &\leq s - k + i \\ &< s - k + n, \end{aligned}$$

since  $t_i + k - 1 \leq s$ . Then

$$\begin{aligned} \Delta^{r-1}G_n(t_r, s) &= \Delta^{r-1}u_{n,i}(t_r, s) \\ &= (-1)^{n+1} \begin{vmatrix} 0 & \Delta^{r-1}y_1(t_r, t_1) & \cdots & \Delta^{r-1}y_{n-1}(t_r, t_1) \\ 0 & 1 & \cdots & y_{n-2}(t_2, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{n-r}(t_r, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & \cdots & 1 \end{vmatrix} \\ &= (-1)^{n+1} \begin{vmatrix} 0 & y_{2-r}(t_r, t_1) & \cdots & y_{n-r}(t_r, t_1) \\ 0 & 1 & \cdots & y_{n-2}(t_2, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{n-r}(t_r, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & \cdots & 1 \end{vmatrix} \\ &= 0 \end{aligned}$$

as the first row and the  $r^{\text{th}}$  row are the same. Therefore,

$$\Delta^{r-1}G_n(t_r, s) = 0$$

for  $r = 2, 3, \dots, i$ . If  $1 \leq i < r \leq n$ , then  $t_{i+1} \leq t_r$ . Hence,

$$t_r \geq t_{i+1} \geq s - k + 1,$$

because  $s \leq t_{i+1} + k - 1$ . As a result,

$$\begin{aligned}
\Delta^{r-1}G_n(t_r, s) &= \Delta^{r-1}v_{n,i}(t_r, s) \\
&= \Delta^{r-1}u_{n,i}(t_r, s) + (-1)^{n-1}\Delta^{r-1}y_{n-1}(t_r, s - k + 1) \\
&= (-1)^{n+1} \begin{vmatrix} 0 & H(2-r) & \cdots & y_{n-r}(t_r, t_1) \\ 0 & 1 & \cdots & y_{n-2}(t_2, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_{i+1}(s, i) & 0 & \cdots & y_{n-(i+1)}(t_{i+1}, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_r(s, i) & 0 & \cdots & y_{n-r}(t_r, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{vmatrix} \\
&\quad + (-1)^{n-1}y_{n-r}(t_r, s - k + 1) \\
&= (-1)^{n+1} \begin{vmatrix} y_{n-r}(t_r, s - k + 1) & H(2-r) & \cdots & y_{n-r}(t_r, t_1) \\ 0 & 1 & \cdots & y_{n-2}(t_2, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_{i+1}(s, i) & 0 & \cdots & y_{n-i-1}(t_{i+1}, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-r}(t_r, s - k + 1) & 0 & \cdots & y_{n-r}(t_r, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{vmatrix}
\end{aligned}$$

by the definition of  $h_r(s, i)$  given in (8). Again, since the first and  $r^{\text{th}}$  rows are equal,

$$\Delta^{r-1}G_n(t_r, s) = 0$$

for  $r = i + 1, i + 2, \dots, n$ . Thus, property (iii) of Lemma 1 is met.  $\square$

## 4 Properties of $u_{n,i}(t, s)$ and $v_{n,i}(t, s)$

We would like to know conditions that are sufficient for the Green's function  $G_n(t, s)$  given in (10) to be positive on  $(t_1, t_n + n - 1] \times [t_1 + k, t_n + k - 1]$ . To do this we will want to know how  $G_n(t, s)$  behaves as a function of both  $t$  and  $s$ . But before we can say something about the difference of  $G_n(t, s)$  with respect to  $t$  and  $s$ , respectively, we need the following lemmas.

**Lemma 3** *Let  $a$  and  $b$  be fixed integers. For any integer  $t$  and any positive integer  $j$ ,*

$$y_j(t, b) = y_j(t, a) - y_j(b, a) - \sum_{m=1}^{j-1} y_{j-m}(b, a)y_m(t, b).$$

*Proof:* By definition,

$$y_j(t, b) = \frac{1}{j!}(t-b)^{(j)} \equiv \frac{1}{j!}(t-b)(t-b-1)\cdots(t-b-j+1).$$

Thus,  $y_j(t, b)$  satisfies

$$\Delta^p y_j(b, b) = 0 \text{ for } p \in \{0, 1, \dots, j-1\},$$

and

$$\Delta^j y_j(t, b) \equiv 1.$$

Set

$$w(t) := y_j(t, a) - y_j(b, a) - \sum_{m=1}^{j-1} y_{j-m}(b, a) y_m(t, b).$$

Then  $w(b) = 0$ , and for  $p \in \{1, 2, \dots, j-1\}$ ,

$$\begin{aligned} \Delta^p w(t) &= \Delta^p y_j(t, a) - \sum_{m=1}^{j-1} y_{j-m}(b, a) \Delta^p y_m(t, b) \\ &= y_{j-p}(t, a) - \sum_{m=p}^{j-1} y_{j-m}(b, a) y_{m-p}(t, b) \end{aligned}$$

by (6), so that when  $t = b$ , we have

$$\begin{aligned} \Delta^p w(b) &= y_{j-p}(b, a) - \sum_{m=p}^{j-1} y_{j-m}(b, a) y_{m-p}(b, b) \\ &= y_{j-p}(b, a) - y_{j-p}(b, a) \cdot 1 \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} \Delta^j w(t) &= \Delta^j y_j(t, a) - \sum_{m=1}^{j-1} y_{j-m}(b, a) \Delta^j y_m(t, b) \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

Therefore  $w(t)$  and  $y_j(t, b)$  satisfy the same conditions, and by uniqueness of solutions, they are equal.  $\square$

**Lemma 4** Consider  $u_{n,i}(t, s : t_1, t_2, \dots, t_n)$  as given in (7). Then we have

$$\Delta_t u_{n,i}(t, s : t_1, t_2, \dots, t_n) = -u_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n)$$

for  $i = 2, 3, \dots, n-1$ , and

$$\Delta_t u_{n,1}(t, s : t_1, t_2, \dots, t_n) = (-1)^n y_{n-2}(t, s - k + 1).$$

*Proof:* Fix an integer  $s$  and think of  $u_{n,i}(t, s) \equiv u_{n,i}(t, s : t_1, t_2, \dots, t_n)$  as a function of  $t$ . Then

$$\Delta_t u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} 0 & 1 & y_1(t, t_1) & \cdots & y_{n-2}(t, t_1) \\ h_2(s, i) & 1 & y_1(t_2, t_1) & \cdots & y_{n-2}(t_2, t_1) \\ h_3(s, i) & 0 & 1 & \cdots & y_{n-3}(t_3, t_1) \\ h_4(s, i) & 0 & 0 & \cdots & y_{n-4}(t_4, t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix},$$

using (5); here  $h_j(s, i)$  are as defined in (8). Replace row 1 of this  $n \times n$  determinant by (row 1 - row 2), and then expand along the 1 in the second row, second column to get the  $n - 1 \times n - 1$  determinant

$$\Delta_t u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} -h_2(s, i) & y_1(t, t_2) & \hat{y}_2(t) & \cdots & \hat{y}_{n-2}(t) \\ h_3(s, i) & 1 & y_1(t_3, t_1) & \cdots & y_{n-3}(t_3, t_1) \\ h_4(s, i) & 0 & 1 & \cdots & y_{n-4}(t_4, t_1) \\ h_5(s, i) & 0 & 0 & \cdots & y_{n-5}(t_5, t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix},$$

where  $\hat{y}_j(t) := y_j(t, t_1) - y_j(t_2, t_1)$  for  $j = 2, 3, \dots, n - 2$ . Now consider the entries in column(3); we want to replace column(3) by

$$\text{column}(3) - y_1(t_2, t_1) \cdot \text{column}(2).$$

For example, replace the first entry  $\hat{y}_2(t)$  by  $\hat{y}_2(t) - y_1(t_2, t_1) \cdot y_1(t, t_2)$ , which is

$$y_2(t, t_1) - y_2(t_2, t_1) - y_1(t_2, t_1)y_1(t, t_2) \tag{12}$$

from the definition of  $\hat{y}_2$ . Using Lemma 3 with  $a = t_1$  and  $b = t_2$ , (12) is  $y_2(t, t_2)$ . Likewise replace  $y_1(t_3, t_1)$  in column(3) by  $y_1(t_3, t_1) - y_1(t_2, t_1)$ , which is  $y_1(t_3, t_2)$ . Thus we have

$$\Delta_t u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} -h_2(s, i) & y_1(t, t_2) & y_2(t, t_2) & \hat{y}_3(t) & \cdots & \hat{y}_{n-2}(t) \\ h_3(s, i) & 1 & y_1(t_3, t_2) & y_2(t_3, t_1) & \cdots & y_{n-3}(t_3, t_1) \\ h_4(s, i) & 0 & 1 & y_1(t_4, t_1) & \cdots & y_{n-4}(t_4, t_1) \\ h_5(s, i) & 0 & 0 & 1 & \cdots & y_{n-5}(t_5, t_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & 0 & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

In the same way, replace column( $j$ ) by

$$\text{column}(j) - \sum_{m=1}^{j-2} y_{j-m-1}(t_2, t_1) \cdot \text{column}(m+1)$$

for  $j = 4, 5, \dots, n-1$ , each time simplifying the result using Lemma 3 with  $a = t_1$  and  $b = t_2$ . This process ends with

$$\Delta_t u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} -h_2(s, i) & y_1(t, t_2) & y_2(t, t_2) & \cdots & y_{n-2}(t, t_2) \\ h_3(s, i) & 1 & y_1(t_3, t_2) & \cdots & y_{n-3}(t_3, t_2) \\ h_4(s, i) & 0 & 1 & \cdots & y_{n-4}(t_4, t_2) \\ h_5(s, i) & 0 & 0 & \cdots & y_{n-5}(t_5, t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & 0 & \cdots & y_1(t_{n-1}, t_2) \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

By adding and subtracting  $H(1-i)y_{n-2}(t, s-k+1)$ , we come up with

$$\Delta_t u_{n,i}(t, s) = (-1)^n H(1-i)y_{n-2}(t, s-k+1)$$

$$+(-1)^{n+1} \begin{vmatrix} \hat{h}(t, s, i) & y_1(t, t_2) & \cdots & y_{n-2}(t, t_2) \\ h_3(s, i) & 1 & \cdots & y_{n-3}(t_3, t_2) \\ h_4(s, i) & 0 & \cdots & y_{n-4}(t_4, t_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(s, i) & 0 & \cdots & y_1(t_{n-1}, t_2) \\ 1 & 0 & \cdots & 1 \end{vmatrix},$$

where

$$\begin{aligned} \hat{h}(t, s, i) &:= H(1-i)y_{n-2}(t, s-k+1) - h_2(s, i) \\ &= H(1-i)[y_{n-2}(t, s-k+1) - y_{n-2}(t_2, s-k+1)]. \end{aligned}$$

This time replace column(1) by

$$\text{column}(1) - H(1-i) \sum_{m=1}^{n-3} y_{n-2-m}(t_2, s-k+1) \cdot \text{column}(m+1). \quad (13)$$

For example, the first entry,  $\hat{h}(t, s, i)$ , is replaced by

$$\hat{h}(t, s, i) - H(1-i) \sum_{m=1}^{n-3} y_{n-2-m}(t_2, s-k+1) y_m(t, t_2),$$

rewritten as

$$\begin{aligned} &H(1-i)[y_{n-2}(t, s-k+1) - y_{n-2}(t_2, s-k+1) \\ &\quad - \sum_{m=1}^{n-3} y_{n-2-m}(t_2, s-k+1) y_m(t, t_2)], \end{aligned}$$

which is

$$H(1-i)y_{n-2}(t, t_2)$$

by Lemma 3, with  $b = t_2$ ,  $a = s - k + 1$ , and  $j = n - 2$ . The overall outcome of (13) is

$$\Delta_t u_{n,i}(t, s) = (-1)^n H(1 - i) y_{n-2}(t, s - k + 1)$$

$$-(-1)^n \begin{vmatrix} H(1 - i) y_{n-2}(t, t_2) & y_1(t, t_2) & \cdots & y_{n-2}(t, t_2) \\ \hat{h}_2(s, i) & 1 & \cdots & y_{n-3}(t_3, t_2) \\ \hat{h}_3(s, i) & 0 & \cdots & y_{n-4}(t_4, t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_{n-2}(s, i) & 0 & \cdots & y_1(t_{n-1}, t_2) \\ 1 & 0 & \cdots & 1 \end{vmatrix} \quad (14)$$

where

$$\begin{aligned} \hat{h}_j(s, i) &:= h_{j+1}(s, i) - H(1 - i) \sum_{m=1}^{n-3} y_{n-2-m}(t_2, s - k - 1) [\text{row}(j), \text{column}(m + 1)] \\ &= h_{j+1}(s, i) - H(1 - i) \sum_{m=j-1}^{n-3} y_{n-2-m}(t_2, s - k - 1) y_{m-j+1}(t_{j+1}, t_2) \end{aligned} \quad (15)$$

for  $j = 2, 3, \dots, n - 2$ ; the second equality follows from the fact that

$$(\text{row}(j), \text{column}(j)) = 1 \quad \text{and} \quad (\text{row}(j), \text{column}(m + 1)) = 0$$

when  $m + 1 < j$  in the determinant in (14). Break off the  $m = j - 1$  term from the sum in (15), and use the definition of  $h_{j+1}(s, i)$  given in (8) to get

$$\begin{aligned} \hat{h}_j(s, i) &= H(j - i) y_{n-j-1}(t_{j+1}, s - k + 1) - H(1 - i) y_{n-j-1}(t_2, s - k + 1) \\ &\quad - H(1 - i) \sum_{m=j}^{n-3} y_{n-2-m}(t_2, s - k - 1) y_{m-j+1}(t_{j+1}, t_2), \end{aligned} \quad (16)$$

for  $j = 2, 3, \dots, n - 2$  and  $i = 1, 2, \dots, n - 1$ .

Now for  $i \in \{2, 3, \dots, n - 1\}$ , we have  $H(1 - i) = 0$ , so that (16) simplifies to

$$\hat{h}_j(s, i) = H(j - 1 - (i - 1)) y_{(n-1)-j}(t_{j+1}, s - k + 1); \quad (17)$$

thus, (14) becomes

$$\Delta_t u_{n,i}(t, s) = -(-1)^n \begin{vmatrix} 0 & y_1(t, t_2) & \cdots & y_{n-2}(t, t_2) \\ \hat{h}_2(s, i) & 1 & \cdots & y_{n-3}(t_3, t_2) \\ \hat{h}_3(s, i) & 0 & \cdots & y_{n-4}(t_4, t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_{n-2}(s, i) & 0 & \cdots & y_1(t_{n-1}, t_2) \\ 1 & 0 & \cdots & 1 \end{vmatrix}, \quad (18)$$

an  $(n - 1) \times (n - 1)$  determinant, for  $j = 2, 3, \dots, n - 2$ . Hence, using (7), (8), (17), and (18), we see that

$$\Delta u_{n,i}(t, s : t_1, t_2, \dots, t_n) = -u_{n-1, i-1}(t, s : t_2, t_3, \dots, t_n),$$

for  $i = 2, 3, \dots, n-1$ . If  $i = 1$ , then (16) becomes

$$\begin{aligned}\hat{h}_j(s, 1) &= y_{n-j-1}(t_{j+1}, s-k+1) - y_{n-j-1}(t_2, s-k+1) \\ &\quad - \sum_{m=j}^{n-3} y_{n-2-m}(t_2, s-k-1)y_{m-j+1}(t_{j+1}, t_2);\end{aligned}$$

replace  $m$  by  $m+j-1$  in the sum to obtain

$$\begin{aligned}\hat{h}_j(s, 1) &= y_{n-j-1}(t_{j+1}, s-k+1) - y_{n-j-1}(t_2, s-k+1) \\ &\quad - \sum_{m=1}^{n-j-2} y_{n-j-1-m}(t_2, s-k-1)y_m(t_{j+1}, t_2)\end{aligned}$$

for  $j = 2, 3, \dots, n-2$ . By Lemma 3, with  $a = s-k+1$ ,  $b = t_2$ ,  $t = t_{j+1}$ , and  $j = n-j-1$ , we get that

$$\hat{h}_j(s, 1) = y_{n-j-1}(t_{j+1}, t_2). \quad (19)$$

Consequently, with  $i = 1$  and (19), (14) becomes

$$\Delta_t u_{n,1}(t, s) = (-1)^n y_{n-2}(t, s-k+1)$$

$$-(-1)^n \begin{vmatrix} y_{n-2}(t, t_2) & y_1(t, t_2) & \cdots & y_{n-2}(t, t_2) \\ y_{n-3}(t_3, t_2) & 1 & \cdots & y_{n-3}(t_3, t_2) \\ y_{n-4}(t_4, t_2) & 0 & \cdots & y_{n-4}(t_4, t_2) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(t_{n-1}, t_2) & 0 & \cdots & y_1(t_{n-1}, t_2) \\ 1 & 0 & \cdots & 1 \end{vmatrix};$$

since the first and last columns of the determinant are the same, the determinant vanishes, leaving

$$\Delta_t u_{n,1}(t, s) = (-1)^n y_{n-2}(t, s-k+1),$$

and the proof is complete.  $\square$

**Lemma 5** Consider  $v_{n,i}(t, s)$  as given in (9). For  $i = 2, 3, \dots, n-1$ ,

$$\Delta_t v_{n,i}(t, s : t_1, t_2, \dots, t_n) = -v_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n),$$

and

$$\Delta_t v_{n,1}(t, s : t_1, t_2, \dots, t_n) = 0.$$

*Proof:* By (9),

$$v_{n,i}(t, s) := u_{n,i}(t, s) + (-1)^{n-1} y_{n-1}(t, s-k+1).$$

Then for  $i \in \{2, 3, \dots, n-1\}$  we have by Lemma 4 and equation (5) that

$$\begin{aligned}\Delta_t v_{n,i}(t, s) &= \Delta_t u_{n,i}(t, s) + (-1)^{n-1} \Delta_t y_{n-1}(t, s-k+1) \\ &= -u_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n) - (-1)^{n-2} y_{n-2}(t, s-k+1) \\ &= -v_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n).\end{aligned}$$

When  $i = 1$ ,

$$\begin{aligned}
\Delta_t v_{n,1}(t, s) &= \Delta_t u_{n,1}(t, s) + (-1)^{n-1} \Delta_t y_{n-1}(t, s - k + 1) \\
&= (-1)^n y_{n-2}(t, s - k + 1) + (-1)^{n-1} y_{n-2}(t, s - k + 1) \\
&= 0.
\end{aligned}$$

□

**Lemma 6** Consider  $y_j(t, s)$  as given in (4). As a function of  $s$ ,

$$\Delta_s y_j(t, s) = -y_{j-1}(t, s + 1).$$

*Proof:* By (4) and (i) in the definition of the factorial function,

$$\begin{aligned}
y_j(t, s) &= \frac{1}{j!} (t - s)^{(j)} \\
&= \frac{1}{j!} (t - s)(t - s - 1) \cdots (t - s - j + 1) \\
&= \frac{1}{j!} (-1)^j (s - t)(s + 1 - t) \cdots (s + j - 1 - t) \\
&= (-1)^j \frac{1}{j!} (s + j - 1 - t)^{(j)} \\
&= (-1)^j y_j(s, t - j + 1).
\end{aligned} \tag{20}$$

Consequently, using (5) with the roles of  $t$  and  $s$  reversed and (20), we have

$$\begin{aligned}
\Delta_s y_j(t, s) &= (-1)^j \Delta_s y_j(s, t - j + 1) \\
&= (-1)^j y_{j-1}(s, t - j + 1) \\
&= (-1)^j (-1)^{j-1} y_{j-1}(t - j + 1, s - (j - 1) + 1) \\
&= -y_{j-1}(t, s + 1).
\end{aligned}$$

□

**Lemma 7** Consider  $u_{n,i}(t, s)$  and  $v_{n,i}(t, s)$  as functions of  $s$ , with  $t \in \mathbb{Z}$  fixed. Then we have the following:

$$\Delta_s u_{n,i}(t, s : t_1, t_2, \dots, t_n) = u_{n-1,i}(t, s + 1 : t_1, t_2, \dots, t_{n-1})$$

for  $i = 1, 2, \dots, n - 2$ , while

$$\Delta_s u_{n,n-1}(t, s : t_1, t_2, \dots, t_n) = 0,$$

and

$$\Delta_s v_{n,i}(t, s : t_1, t_2, \dots, t_n) = v_{n-1,i}(t, s + 1 : t_1, t_2, \dots, t_{n-1})$$

for  $i = 1, 2, \dots, n - 2$ , with

$$\Delta_s v_{n,n-1}(t, s : t_1, t_2, \dots, t_n) = y_{n-2}(s - k + n - 1, t).$$

*Proof:* Recall that  $u_{n,i}(t, s)$  is defined in terms of the  $n \times n$  determinant

$$u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} 0 & y_1(t, t_1) & y_2(t, t_1) & \cdots & y_{n-1}(t, t_1) \\ h_2(n, s, i) & 1 & y_1(t_2, t_1) & \cdots & y_{n-2}(t_2, t_1) \\ h_3(n, s, i) & 0 & 1 & \cdots & y_{n-3}(t_3, t_1) \\ h_4(n, s, i) & 0 & 0 & \cdots & y_{n-4}(t_4, t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1}(n, s, i) & 0 & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix},$$

with

$$h_j(n, s, i) := H(j-1-i)y_{n-j}(t_j, s-k+1) \quad (21)$$

for  $j = 2, 3, \dots, n-1$  and  $i = 1, 2, \dots, n-1$ . By Lemma 6,

$$\begin{aligned} \Delta_s h_j(n, s, i) &= -H(j-1-i)y_{n-j-1}(t_j, s+1-k+1) \\ &= -H(j-1-i)y_{(n-1)-j}(t_j, (s+1)-k+1) \\ &= -h_j(n-1, s+1, i), \end{aligned}$$

where the second equality follows from the definition of  $y_j$  in (4). As a result,

$$\Delta_s u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} 0 & y_1(t, t_1) & \cdots & y_{n-1}(t, t_1) \\ -h_2(n-1, s+1, i) & 1 & \cdots & y_{n-2}(t_2, t_1) \\ -h_3(n-1, s+1, i) & 0 & \cdots & y_{n-3}(t_3, t_1) \\ -h_4(n-1, s+1, i) & 0 & \cdots & y_{n-4}(t_4, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ -h_{n-1}(n-1, s+1, i) & 0 & \cdots & y_1(t_{n-1}, t_1) \\ 0 & 0 & \cdots & 1 \end{vmatrix}.$$

We factor the  $-1$  out of the first column and expand along the last row to obtain the  $(n-1) \times (n-1)$  determinant

$$\Delta_s u_{n,i}(t, s) = (-1)^n \begin{vmatrix} 0 & y_1(t, t_1) & \cdots & y_{n-2}(t, t_1) \\ h_2(n-1, s+1, i) & 1 & \cdots & y_{n-3}(t_2, t_1) \\ h_3(n-1, s+1, i) & 0 & \cdots & y_{n-4}(t_3, t_1) \\ h_4(n-1, s+1, i) & 0 & \cdots & y_{n-5}(t_4, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-2}(n-1, s+1, i) & 0 & \cdots & y_1(t_{n-2}, t_1) \\ h_{n-1}(n-1, s+1, i) & 0 & \cdots & 1 \end{vmatrix}. \quad (22)$$

Consider the lower left entry in the determinant in (22). By (21),

$$h_{n-1}(n-1, s+1, i) = H(n-2-i)y_{(n-1)-(n-1)}(t_{n-1}, s+1-k+1).$$

For  $i = 1, 2, \dots, n-2$ ,  $H(n-2-i) = 1$ , so that by convention

$$h_{n-1}(n-1, s+1, i) = 1,$$

and we have

$$\Delta_s u_{n,i}(t, s) = u_{n-1,i}(t, s+1).$$

If  $i = n - 1$ , then

$$H(j - 1 - (n - 1)) = H(j - n) = 0$$

for all  $j \in \{2, 3, \dots, n - 1\}$ , whence (21) yields

$$h_j(n, s, n - 1) = 0.$$

This zeros out the first column in (22), and we have

$$\Delta_s u_{n,n-1}(t, s) \equiv 0.$$

Now consider  $v_{n,i}(t, s)$ . For  $i = 1, 2, \dots, n - 2$  we have by (9), Lemma 6, and the earlier part of this proof that

$$\begin{aligned} \Delta_s v_{n,i}(t, s) &= \Delta_s u_{n,i}(t, s) + (-1)^{n-1} \Delta_s y_{n-1}(t, s - k + 1) \\ &= u_{n-1,i}(t, s + 1) + (-1)^{(n-1)-1} y_{(n-1)-1}(t, (s + 1) - k + 1) \\ &= v_{n-1,i}(t, s + 1). \end{aligned}$$

When  $i = n - 1$ , then

$$\begin{aligned} \Delta_s v_{n,n-1}(t, s) &= \Delta_s u_{n,n-1}(t, s) + (-1)^{n-1} \Delta_s y_{n-1}(t, s - k + 1) \\ &= 0 + (-1)^{n-2} y_{n-2}(t, s - k + 2) \\ &= y_{n-2}(s - k + n - 1, t) \end{aligned}$$

by (20). □

## 5 Differences of $G_n(t, s)$ with respect to $t$ and $s$

**Theorem 8** Consider Green's function  $G_n(t, s : t_1, t_2, \dots, t_n)$  given in (10) as a function of both  $t$  and  $s$ . Then, for  $n \geq 3$ , we have

$$\Delta_t G_n(t, s : t_1, \dots, t_n) = \begin{cases} s \in I_1 : \begin{cases} y_{n-2}(s - k + n - 2, t) & : t < s - k + n - 1 \\ 0 & : t \geq s - k + 1 \end{cases} \\ -G_{n-1}(t, s : t_2, t_3, \dots, t_n) \begin{cases} t \in \mathbb{Z} \\ s \in [t_2 + k - 1, t_n + k - 1] \end{cases} \end{cases}$$

and

$$\Delta_s G_n(t, s : t_1, \dots, t_n) = \begin{cases} G_{n-1}(t, s + 1 : t_1, \dots, t_{n-1}) \begin{cases} t \in \mathbb{Z} \\ s \in [t_1 + k, t_{n-1} + k - 1] \end{cases} \\ s \in I_{n-1} : \begin{cases} 0 & : t < s - k + n - 1 \\ y_{n-2}(s - k + n - 1, t) & : t \geq s - k + 1. \end{cases} \end{cases}$$

*Proof:* For  $s \in I_1$ , we have by Lemmas 4 and 5 and equation (20) that

$$\begin{cases} \Delta_t u_{n,1}(t, s : t_1, \dots, t_n) \\ \Delta_t v_{n,1}(t, s : t_1, \dots, t_n) \end{cases} = \begin{cases} y_{n-2}(s - k + n - 2, t) \\ 0. \end{cases}$$

For  $s \in I_i$ , we again use Lemmas 4 and 5 to get

$$\begin{cases} \Delta_t u_{n,i}(t, s : t_1, t_2, \dots, t_n) \\ \Delta_t v_{n,i}(t, s : t_1, t_2, \dots, t_n) \end{cases} = \begin{cases} -u_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n) \\ -v_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n), \end{cases}$$

for  $i = 2, 3, \dots, n - 1$ .

Now fix  $t$  and think of  $G_n(t, s : t_1, t_2, \dots, t_n)$  as a function of  $s$ . Then Lemma 7 yields

$$\begin{cases} \Delta_s u_{n,i}(t, s) \\ \Delta_s v_{n,i}(t, s) \end{cases} = \begin{cases} u_{n-1,i}(t, s + 1) \\ v_{n-1,i}(t, s + 1) \end{cases}$$

for  $i = 1, 2, \dots, n - 2$ , and

$$\begin{cases} \Delta_s u_{n,n-1}(t, s) \\ \Delta_s v_{n,n-1}(t, s) \end{cases} = \begin{cases} 0 \\ y_{n-2}(s - k + n - 1, t) \end{cases}$$

when  $i = n - 1$ . □

## 6 Positivity of the Green's Function

**Lemma 9** For  $s \in I_1 = [t_1 + k, t_2 + k - 1]$  and  $n \geq 2$ ,

$$G_n(t, s : t_1, t_2, \dots, t_n) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_n + n - 1]. \end{cases} \quad C(n)$$

*Proof:* Note that by (7), (9), and (10) with  $n = 2$ ,

$$G_2(t, s : t_1, t_2) = \begin{cases} t - t_1 & : t < s - k + 2 \\ s - k + 1 - t_1 & : t \geq s - k + 1, \end{cases} \quad (23)$$

so that  $C(2)$  holds. When  $n \geq 3$  and  $t \in (-\infty, s - k + 1)$ ,

$$\Delta_t G_n(t, s) = y_{n-2}(s - k + n - 2, t)$$

by Theorem 8, whence  $\Delta_t G_n(t, s) > 0$  for all  $t \in (-\infty, s - k + 1)$ . Since  $G_n(t_1, s) = 0$  by (iii) of Lemma 1,

$$G_n(t, s : t_1, t_2, \dots, t_n) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, s - k + 1). \end{cases}$$

But as  $\Delta G_n(t, s) \equiv 0$  for  $t \geq s - k + 1$  from Theorem 8, we have  $G_n(t, s) > 0$  on  $[s - k + 1, t_n + n - 1]$  as well. Hence  $C(n)$  also holds for  $n \geq 3$ . □

**Theorem 10** Assume for  $n \geq 4$  that

$$v_{n-i,j-i+1}(t_n + n - i, s_j + 1 : t_i, t_{i+1}, \dots, t_{n-1}) > 0 \quad H_1(n)$$

for  $j \in \{2, 3, \dots, n-2\}$  and  $i = j-1, j-2, \dots, 1$ , and for  $s_j \in [t_j + k - 1, t_{j+1} + k - 1]$ .  
Then

$$G_n(t, s : t_1, t_2, \dots, t_n) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_n + n - 1] \end{cases} \quad C(n)$$

for  $s \in [t_1 + k, t_n + k - 1]$  if  $n$  is even, or for  $s \in [t_1 + k, t_{n-1} + k - 1]$  if  $n$  is odd. For odd  $n \geq 3$ , the additional assumption

$$u_{n,n-1}(t_n + n - 1) > -1 \quad H_2(n)$$

yields  $C(n)$  for  $s \in [t_{n-1} + k - 1, t_n + k - 1]$  as well.

*Proof:* We proceed by induction on  $n \geq 3$ : For  $n = 3$ , we assume  $H_2(3)$  holds, and consider  $G_3(t, s : t_1, t_2, t_3)$  for  $t \in [t_1, t_3 + 2]$  and  $s \in [t_1 + k, t_3 + k - 1]$ . Note that by (23),

$$G_2(t, s : t_1, t_2) < 0$$

on  $(-\infty, t_1)$  and

$$G_2(t, s : t_1, t_2) > 0$$

on  $(t_1, t_3 + 1]$ , with  $s \in [t_1 + k, t_2 + k - 1]$ . If  $s \in I_1$ , we have that  $C(3)$  holds by Lemma 9. Next, let  $s \in I_2 = [t_2 + k - 1, t_3 + k - 1]$ . We have

$$\Delta G_3(t, s) = -G_2(t, s : t_2, t_3) > 0$$

on  $(-\infty, t_2)$ , after using the note above and Theorem 8. Thus,  $G_3(t, s) > 0$  on  $(t_1, t_2 + 1]$ . For  $t \in (t_2, t_3 + 1]$ ,

$$\Delta G_3(t, s) = -G_2(t, s : t_2, t_3)$$

implies that  $\Delta G_3(t, s) < 0$  on  $(t_2, t_3 + 1]$ . Hence,  $G_3$  is decreasing in  $t$  on  $(t_2, t_3 + 2]$ , so that if  $G_3(t_3 + 2, s) > 0$ , then  $G_3(t, s) > 0$  for all  $t \in (t_1, t_3 + 2]$ . Now by Theorem 8,

$$\begin{aligned} \Delta_s G_3(t_3 + 2, s) &= y_1(s - k + 2, t_3 + 2) \\ &= s - k - t_3 \\ &< 0, \end{aligned}$$

since  $s \leq t_3 + k - 1$ . Consequently,  $\Delta_s G_3(t_3 + 2, s) < 0$  for all  $s \in I_2$ . In other words,

$$G_3(t_3 + 2, t_3 + k - 1) \leq G_3(t_3 + 2, s) \quad (24)$$

for  $s \in [t_2 + k - 1, t_3 + k - 1]$ . Yet

$$\begin{aligned} G_3(t_3 + 2, t_3 + k - 1) &= u_{3,2}(t_3 + 2) + y_2(t_3 + 2, t_3) \\ &= u_{3,2}(t_3 + 2) + 1 \\ &> 0, \end{aligned}$$

as  $u_{3,2}(t_3 + 2) > -1$  by  $H_2(3)$ . Therefore  $G_3(t, s) > 0$  on  $(t_1, t_3 + 2]$  for  $s \in [t_1 + k, t_3 + k - 1]$ .

Now assume that  $H_1(n-1)$  and  $H_2(n-1)$  (if necessary) imply that  $C(n-1)$  holds for  $s \in [t_1 + k, t_{n-1} + k - 1]$ . We will show that  $H_1(n)$  and  $H_2(n)$  (if  $n$  odd) yield  $C(n)$  for  $s \in [t_1 + k, t_n + k - 1]$ . Hence, suppose  $H_1(n)$  is true.

Case (1): For  $s \in I_1$ , Lemma 9 gives  $C(n)$ .

Case (2): Let  $s \in I_2$ . We have from Theorem 8 that

$$\Delta_t G_n(t, s : t_1, t_2, \dots, t_n) = -G_{n-1}(t, s : t_2, t_3, \dots, t_n).$$

By Lemma 9 then,

$$\Delta_t G_n(t, s : t_1, t_2, \dots, t_n) \begin{cases} > 0 & : t \in (-\infty, t_2) \\ < 0 & : t \in (t_2, t_n + n - 2]. \end{cases}$$

Again, since  $G_n(t_1, s) = 0$ ,

$$G_n(t, s : t_1, t_2, \dots, t_n) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_2 + 1]. \end{cases}$$

For  $t \in (t_2, t_n + n - 1]$ ,  $G_n(t, s)$  is decreasing in  $t$ , so if  $G_n(t_n + n - 1, s) > 0$  for all  $s \in I_2$ , then  $G_n(t, s) > 0$  for all  $t \in (t_1, t_n + n - 1]$  and  $s \in I_2$ . Thinking of  $G_n(t_n + n - 1, s : t_1, t_2, \dots, t_n)$  as a function of  $s$ , Theorem 8 gives that

$$\begin{aligned} \Delta_s G_n(t_n + n - 1, s : t_1, \dots, t_n) &= G_{n-1}(t_n + n - 1, s + 1 : t_1, \dots, t_{n-1}) \\ &= v_{n-1,2}(t_n + n - 1, s + 1 : t_1, \dots, t_{n-1}) \\ &> 0 \end{aligned}$$

by  $H_1(n)$  with  $j = 2$  and  $i = 1$ . Therefore  $G_n(t_n + n - 1, s)$  is increasing in  $s$ ; in other words,

$$G_n(t_n + n - 1, t_2 + k - 1) \leq G_n(t, s)$$

for all  $t \in (t_2, t_n + n - 1]$  and  $s \in I_2$ . But by Lemma 9,

$$G_n(t_n + n - 1, t_2 + k - 1) > 0.$$

Thus

$$G_n(t, s) > 0 \tag{25}$$

on  $(t_1, t_n + n - 1]$  for  $s \in [t_2 + k - 1, t_3 + k - 1]$ .

Case (j): Let  $s_j \in I_j = [t_j + k - 1, t_{j+1} + k - 1]$ ,  $j \in \{3, 4, \dots, n - 2\}$ . As before we use Theorem 8 to get

$$\Delta_t G_n(t, s : t_1, t_2, \dots, t_n) = -G_{n-1}(t, s + 1 : t_2, t_3, \dots, t_n). \tag{26}$$

Recall by the induction hypothesis that  $C(n-1)$ , or

$$G_{n-1}(t, s : t_1, t_2, \dots, t_{n-1}) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_{n-1} + n - 2] \end{cases}$$

follows if  $H_1(n-1)$ , or

$$v_{n-1-i, j-i+1}(t_{n-1} + n - 1 - i, s_j + 1 : t_i, t_{i+1}, \dots, t_{n-2}) > 0$$

holds for  $j \in \{2, 3, \dots, n-3\}$ ,  $i = j-1, j-2, \dots, 1$ , and  $s_j \in I_j$ . Consider the hypothesis  $H_1(n)$ ; either  $i = 1$  and

$$v_{n-1,j}(t_n + n - 1, s_j + 1 : t_1, t_2, \dots, t_{n-1}) > 0$$

for  $j \in \{2, 3, \dots, n-2\}$  and  $s_j \in I_j$ , or

$$v_{n-i,j-i+1}(t_n + n - i, s_j + 1 : t_i, t_{i+1}, \dots, t_{n-1}) > 0 \quad (27)$$

for  $j \in \{3, 4, \dots, n-2\}$ ,  $i = j-1, j-2, \dots, 2$ , and  $s_j \in I_j$ . Let  $j \rightarrow J+1$  and  $i \rightarrow I+1$  in (27); then

$$v_{n-1-I,J-I+1}(t_{n-1} + n - 1 - I, s_{J+1} + 1 : t_{I+1}, t_{I+2}, \dots, t_{n-1}) > 0 \quad (28)$$

for  $J \in \{2, 3, \dots, n-3\}$ ,  $I = J-1, J-2, \dots, 1$ , and  $s_{J+1} \in I_{J+1}$ . By the induction hypothesis, (28) implies that

$$-G_{n-1}(t, s_{J+1} : t_2, t_3, \dots, t_n) \begin{cases} > 0 & : t \in (-\infty, t_2) \\ < 0 & : t \in (t_2, t_n + n - 2] \end{cases}$$

for  $s_{J+1} \in [t_{J+1} + k - 1, t_{J+2} + k - 1]$ ,  $J \in \{2, 3, \dots, n-3\}$ . Replacing  $J+1$  by  $j$ , we see that

$$-G_{n-1}(t, s_j : t_2, t_3, \dots, t_n) \begin{cases} > 0 & : t \in (-\infty, t_2) \\ < 0 & : t \in (t_2, t_n + n - 2] \end{cases} \quad (29)$$

for  $s_j \in I_j$  and  $j \in \{3, 4, \dots, n-2\}$ . Consequently,

$$\Delta_t G_n(t, s_j : t_1, t_2, \dots, t_n) \begin{cases} > 0 & : t \in (-\infty, t_2) \\ < 0 & : t \in (t_2, t_n + n - 2] \end{cases}$$

by (26) and (29). As  $G_n(t_1, s_j) = 0$ ,  $G_n(t, s_j) > 0$  for  $t \in (t_1, t_2 + 1]$  and  $s_j \in I_j$ . For  $t \in (t_2, t_n + n - 2]$  and  $s_j \in I_j$ ,

$$\Delta_t G_n(t, s_j) < 0$$

implies that  $G_n$  is decreasing in  $t$ , so if

$$G_n(t_n + n - 1, s_j) > 0,$$

then

$$G_n(t, s_j) > 0$$

for all  $t \in (t_1, t_n + n - 2]$  and  $s_j \in I_j$ . By Theorem 8,

$$\begin{aligned} \Delta_{s_j} G_n(t_n + n - 1, s_j) &= G_{n-1}(t_n + n - 1, s_j + 1 : t_1, \dots, t_{n-1}) \\ &= v_{n-1,j}(t_n + n - 1, s_j + 1 : t_1, \dots, t_{n-1}) \\ &> 0 \end{aligned}$$

for  $j = 3, 4, \dots, n-2$  by  $H_1(n)$ , with  $i = 1$ . Therefore  $G_n(t_n + n - 1, s_j)$  is increasing in  $s_j$ ; in other words,

$$G_n(t_n + n - 1, t_j + k - 1) \leq G_n(t_n + n - 1, s_j) \quad (30)$$

for all  $s_j \in [t_j + k - 1, t_{j+1} + k - 1]$ , for  $j = 3, 4, \dots, n - 2$ . Considering (25), we established in particular that

$$G_n(t_n + n - 1, t_3 + k - 1) > 0. \quad (31)$$

Hence

$$G_n(t_n + n - 1, s_3) > 0$$

for all  $s_3 \in [t_3 + k - 1, t_4 + k - 1]$  by (30) and (31). Again, we specifically have by (30) that

$$0 < G_n(t_n + n - 1, t_4 + k - 1) \leq G_n(t_n + n - 1, s_4)$$

for all  $s_4 \in [t_4 + k - 1, t_5 + k - 1]$ , and so on. Finally, we have that

$$0 < G_n(t_n + n - 1, t_{n-2} + k - 1) \leq G_n(t_n + n - 1, s_{n-2})$$

for all  $s \in [t_{n-2} + k - 1, t_{n-1} + k - 1]$ . Altogether then,

$$G_n(t, s : t_1, t_2, \dots, t_n) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_n + n - 1] \end{cases} \quad (32)$$

for  $s \in [t_3 + k - 1, t_{n-1} + k - 1]$ , for  $n$  both even and odd.

Case  $(n-1)$ : Let  $s \in [t_{n-1} + k - 1, t_n + k - 1]$ .

(i) Assume  $n$  is even. When  $s = t_{n-1} + k - 1$ , we have

$$G_n(t, t_{n-1} + k - 1) = u_{n,n-1}(t) < 0$$

on  $(-\infty, t_1)$ , and

$$0 < G_n(t, t_{n-1} + k - 1) = \begin{cases} u_{n,n-1}(t) & : t < t_{n-1} + n - 1 \\ u_{n,n-1}(t) - y_{n-1}(t, t_{n-1}) & : t \geq t_{n-1} \end{cases}$$

for all  $t \in (t_1, t_n + n - 1]$  by (32). The function  $u_{n,n-1}(t)$  is independent of  $s$ , so actually

$$u_{n,n-1}(t) \begin{cases} < 0 & : t < t_1 \\ > 0 & : t \in (t_1, t_{n-1} + n - 1) \end{cases}$$

for all  $s \in I_{n-1}$ . Moreover,

$$u_{n,n-1}(t) - y_{n-1}(t, t_{n-1}) > 0 \quad (33)$$

for all  $t \in [t_{n-1}, t_n + n - 1]$  means that

$$u_{n,n-1}(t) > y_{n-1}(t, t_{n-1}) \geq 0$$

for all  $t \in [t_{n-1}, t_n + n - 1]$  by the definition of  $y_{n-1}(t, s)$  in (4). Consequently,

$$u_{n,n-1}(t) \begin{cases} < 0 & : t < t_1 \\ > 0 & : t \in (t_1, t_n + n - 1] \end{cases} \quad (34)$$

for all  $s \in I_{n-1}$ . For a fixed  $s \in [t_{n-1} + k - 1, t_n + k - 1]$  and a fixed  $t \in [s - k + 1, t_n + n - 1]$ ,

$$\begin{aligned} G_n(t, s) &= u_{n,n-1}(t) - y_{n-1}(t, s - k + 1) \\ &\geq u_{n,n-1}(t) - y_{n-1}(t, t_{n-1}) \end{aligned}$$

because

$$t \geq s - k + 1 \geq t_{n-1}.$$

Thus,  $G_n(t, s) > 0$  by (33). Since  $t, s$  were arbitrary,

$$G_n(t, s) > 0 \tag{35}$$

for  $t \in (t_1, t_n + n - 1]$  and  $s \in I_{n-1}$ , so that  $C(n)$  follows from (34) and (35).

(ii) Assume  $n$  is odd. Again using Theorem 8, we have

$$\Delta_t G_n(t, s_{n-1} : t_1, t_2, \dots, t_n) = -G_{n-1}(t, s_{n-1} : t_2, t_3, \dots, t_n)$$

for  $s_{n-1} \in I_{n-1} = [t_{n-1} + k - 1, t_n + k - 1]$ . But  $C(n - 1)$ , that is

$$G_{n-1}(t, s_{n-2} : t_1, t_2, \dots, t_{n-1}) \begin{cases} < 0 & : t \in (-\infty, t_1), \\ > 0 & : t \in (t_1, t_{n-1} + n - 2], \end{cases}$$

holds when  $s_{n-2} \in I_{n-2}$  by the earlier part of the theorem if

$$v_{n-2, n-3}(t_{n-1} + n - 2, s_{n-3} + 1 : t_1, t_2, \dots, t_{n-2}) > 0,$$

which is  $H_1(n - 1)$  with  $i = 1$  and  $j = n - 3$ . Thus,

$$G_{n-1}(t, s_{n-1} : t_2, t_3, \dots, t_n) \begin{cases} < 0 & : t \in (-\infty, t_2), \\ > 0 & : t \in (t_2, t_n + n - 2] \end{cases}$$

holds when  $s_{n-1} \in I_{n-1}$  if

$$v_{n-2, n-3}(t_n + n - 2, s_{n-2} + 1 : t_2, t_3, \dots, t_{n-1}) > 0,$$

which is  $H_1(n)$  with  $i = 2$  and  $j = n - 2$ . Because  $H_1(n)$  with  $i = 2$  and  $j = n - 2$  holds, we have

$$\Delta_t G_n(t, s) \begin{cases} > 0 & : t \in (-\infty, t_2), \\ < 0 & : t \in (t_2, t_n + n - 2]. \end{cases} \tag{36}$$

Both  $G_n(t_1, s) = 0$  and (36) imply that

$$G_n(t, s) \begin{cases} < 0 & : t \in (-\infty, t_1), \\ > 0 & : t \in (t_1, t_2 + 1], \end{cases}$$

and that  $G_n(t, s)$  is decreasing in  $t$  over  $(t_2, t_n + n - 1]$  for  $s \in I_{n-1}$ . Hence, if  $G_n(t_n + n - 1, s) > 0$ , then  $G_n(t, s) > 0$  for all  $t \in (t_1, t_n + n - 1]$  and  $s \in I_{n-1}$ . Considering  $G_n(t, s)$  as a function of  $s$  with  $t = t_n + n - 1$ , we have by Theorem 8 that

$$\begin{aligned} \Delta_s G_n(t_n + n - 1, s) &= y_{n-2}(s - k + n - 1, t_n + n - 1) \\ &= y_{n-2}(s - k, t_n) \\ &= \frac{1}{(n-2)!} (s - k - t_n)^{(n-2)} \\ &< 0 \end{aligned}$$

since  $s \leq t_n + k - 1$  and  $n$  is odd. Therefore,  $G_n(t_n + n - 1, s)$  is decreasing in  $s$ , and

$$G_n(t_n + n - 1, t_n + k - 1) \leq G_n(t, s)$$

for all  $t \in (t_2, t_n + n - 1]$  and for all  $s \in [t_{n-1} + k - 1, t_n + k - 1]$ . But

$$\begin{aligned} G_n(t_n + n - 1, t_n + k - 1) &= u_{n,n-1}(t_n + n - 1) + y_{n-1}(t_n + n - 1, t_n) \\ &= u_{n,n-1}(t_n + n - 1) + 1 \\ &> 0 \end{aligned}$$

since  $H_2(n)$  holds for odd  $n$ . □

The following corollaries demonstrate that hypotheses  $H_1(n)$  and  $H_2(n)$ —when  $n$  is odd—are really conditions on the distances between various boundary points. Diaz [2] established the results for  $n = 3$  and  $n = 4$ ; here we employ a different method of proof that reduces the number of assumptions needed in [2] for the  $n = 4$  case.

**Corollary 11** *If*

$$t_2 - t_1 \geq t_3 - t_2 + 1, \tag{37}$$

*then*

$$G_3(t, s : t_1, t_2, t_3) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_3 + 2] \end{cases} \tag{C(3)}$$

for  $s \in [t_1 + k, t_3 + k - 1]$ .

*Proof:* By Theorem 10,  $C(3)$  follows if  $H_2(3)$ , that is  $u_{3,2}(t_3 + 2) > -1$ , holds. Note that

$$\begin{aligned} u_{3,2}(t) &= y_1(t, t_1)y_1(t_2, t_1) - y_2(t, t_1) \\ &= \frac{1}{2}(t - t_1)(2t_2 - t_1 - t + 1) \end{aligned} \tag{38}$$

using (7) and (4). So,  $u_{3,2}(t)$  is a concave-down parabola with zeros at  $t = t_1$  and  $t = 2t_2 - t_1 + 1$ . Thus, if  $t_3 + 2 \leq 2t_2 - t_1 + 1$ , then  $u_{3,2}(t_3 + 2) \geq 0$ ; but this follows from (37). Hence, we have  $C(3)$ . □

**Corollary 12** *If*

$$t_2 - t_1 \geq t_4 - t_2 + 2, \tag{39}$$

*then*

$$G_4(t, s : t_1, t_2, t_3, t_4) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_4 + 3] \end{cases} \tag{C(4)}$$

for  $s \in [t_1 + k, t_4 + k - 1]$ .

*Proof:* Again by Theorem 10,  $C(4)$  follows if  $H_1(4)$ , that is

$$v_{3,2}(t_4 + 3, s + 1 : t_1, t_2, t_3) > 0,$$

holds for  $s \in [t_2 + k - 1, t_3 + k - 1]$ . Thus, let  $s \in I_2$ . Then

$$\begin{aligned} v_{3,2}(t_4 + 3, s + 1 : t_1, t_2, t_3) &= u_{3,2}(t_4 + 3, s + 1) + y_2(t_4 + 3, s - k + 2) \\ &= u_{3,2}(t_4 + 3) + y_2(t_4 + 1, s - k), \end{aligned} \quad (40)$$

as  $u_{n,n-1}$  is independent of  $s$  by Lemma 7. Using (38) we have

$$u_{3,2}(t_4 + 3) \geq 0 \quad (41)$$

if  $t_4 + 3 \leq 2t_2 - t_1 + 1$ , which holds by assumption (39). Since

$$\begin{aligned} y_2(t_4 + 1, s - k) &\geq y_2(t_4 + 1, t_3 - 1) \\ &= \frac{1}{2}(t_4 - t_3 + 2)(t_4 - t_3 + 1) \\ &> 0 \end{aligned}$$

for  $s \in [t_2 + k - 1, t_3 + k - 1]$ ,  $H_1(4)$  holds by (40) and (41).  $\square$

**Corollary 13** *If*

$$t_2 - t_1 \geq t_5 - t_2 + 3, \quad (42)$$

$$t_3 - t_2 \geq t_5 - t_3 + 2, \quad (43)$$

and

$$\begin{vmatrix} y_1(t_5 + 4, t_1) & y_2(t_5 + 4, t_1) & y_3(t_5 + 4, t_1) & y_4(t_5 + 4, t_1) \\ 1 & y_1(t_2, t_1) & y_2(t_2, t_1) & y_3(t_2, t_1) \\ 0 & 1 & y_1(t_3, t_1) & y_2(t_3, t_1) \\ 0 & 0 & 1 & y_1(t_4, t_1) \end{vmatrix} > -1, \quad (44)$$

then

$$G_5(t, s : t_1, \dots, t_5) \begin{cases} < 0 & : t \in (-\infty, t_1) \\ > 0 & : t \in (t_1, t_5 + 4] \end{cases} \quad C(5)$$

for  $s \in [t_1 + k, t_5 + k - 1]$ .

*Proof:* Once more by Theorem 10,  $C(5)$  follows if  $H_1(5)$ , that is

$$v_{4,2}(t_5 + 4, s_2 + 1 : t_1, t_2, t_3, t_4) > 0 \quad (45)$$

for  $s_2 \in [t_2 + k - 1, t_3 + k - 1]$ ,

$$v_{3,2}(t_5 + 3, s_3 + 1 : t_2, t_3, t_4) > 0 \quad (46)$$

and

$$v_{4,3}(t_5 + 4, s_3 + 1 : t_1, t_2, t_3, t_4) > 0 \quad (47)$$

for  $s_3 \in [t_3 + k - 1, t_4 + k - 1]$ , and  $H_2(5)$  (since  $n = 5$  is odd), that is

$$u_{5,4}(t_5 + 4) > -1, \quad (48)$$

hold. To show that (45) and (47) hold, note that

$$v_{4,i}(t_5 + 4, s_i + 1 : t_1, t_2, t_3, t_4) = G_4(t, s_i + 1)$$

for  $i = 2, 3$ . By Corollary 12,  $G_4(t, s) > 0$  for  $t \in (t_1, t_4 + 3]$  and  $s \in [t_1 + k, t_4 + k - 1]$  if

$$t_4 + 3 \leq 2t_2 - t_1 + 1.$$

In the same way,  $G_4(t, s) > 0$  for  $t \in (t_1, t_5 + 4]$  and  $s \in [t_1 + k, t_4 + k - 1]$  if

$$t_5 + 4 \leq 2t_2 - t_1 + 1,$$

which can be rewritten as (42). Hence,

$$G_4(t_5 + 4, s_i + 1) > 0$$

for  $s_i \in [t_i + k - 1, t_{i+1} + k - 1]$  with  $i \in \{2, 3\}$ , so that (45) and (47) are established. Similarly,

$$v_{3,2}(t_5 + 3, s_3 + 1 : t_2, t_3, t_4) = G_3(t_5 + 3, s_3 + 1 : t_2, t_3, t_4).$$

We have from Corollary 11 that  $G_3(t, s : t_1, t_2, t_3) > 0$  for  $t \in (t_1, t_3 + 2]$  and  $s \in [t_1 + k, t_3 + k - 1]$  as long as

$$t_2 - t_1 \geq t_3 - t_2 + 1.$$

With shifted boundary points, this implies that  $G_3(t, s : t_2, t_3, t_4) > 0$  for  $t \in (t_2, t_4 + 2]$  and  $s \in [t_2 + k, t_4 + k - 1]$  as long as

$$t_3 - t_2 \geq t_4 - t_3 + 1,$$

in other words

$$t_3 - t_2 \geq (t_4 + 2) - t_3 - 1.$$

Thus, if we extend the righthand endpoint of the  $t$  interval from  $t_4 + 2$  to  $t_5 + 3$ , we have  $G_3(t, s : t_2, t_3, t_4) > 0$  for  $t \in (t_2, t_5 + 3]$  and  $s \in [t_2 + k, t_4 + k - 1]$  if

$$t_3 - t_2 \geq (t_5 + 3) - t_3 - 1,$$

which is (43). In particular,

$$G_3(t_5 + 3, s_3 + 1 : t_2, t_3, t_4) > 0$$

for  $s_3 \in [t_3 + k, t_4 + k - 1]$ , and (46) follows. Lastly, notice that

$$\begin{aligned} u_{5,4}(t_5 + 4) &= \begin{vmatrix} 0 & y_1(t_5 + 4, t_1) & y_2(t_5 + 4, t_1) & y_3(t_5 + 4, t_1) & y_4(t_5 + 4, t_1) \\ 0 & 1 & y_1(t_2, t_1) & y_2(t_2, t_1) & y_3(t_2, t_1) \\ 0 & 0 & 1 & y_1(t_3, t_1) & y_2(t_3, t_1) \\ 0 & 0 & 0 & 1 & y_1(t_4, t_1) \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} y_1(t_5 + 4, t_1) & y_2(t_5 + 4, t_1) & y_3(t_5 + 4, t_1) & y_4(t_5 + 4, t_1) \\ 1 & y_1(t_2, t_1) & y_2(t_2, t_1) & y_3(t_2, t_1) \\ 0 & 1 & y_1(t_3, t_1) & y_2(t_3, t_1) \\ 0 & 0 & 1 & y_1(t_4, t_1) \end{vmatrix} \\ &> -1 \end{aligned}$$

by(44), so that (48) is satisfied.  $\square$

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