Cost-Conscious Voters in Referendum Elections

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Imagine this...

- Vote on a 3 proposal set
  - Proposal 1: Spend $200,000 to plant new flowers and trees at the local zoo to beautify the scenery
  - Proposal 2: Spend $400,000 to add a kids' play zone
  - Proposal 3: Spend $500,000 to add a brand new panda exhibit

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- You are a cost-conscious voter
  - For instance, you wouldn't want proposals 1 and 2 to both pass if it means spending more than you are willing to spend

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- As a voter, you support the goals of each of the proposals
- You are a cost-conscious voter
  - For instance, you wouldn't want proposals 1 and 2 to both pass if it means spending more than you are willing to spend
- How will you vote?

Cost-Conscious Voters in Referendum Elections

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- When cost-conscious voters are present, can we be assured that there is an outcome that represents the preferences of the electorate?

The Problem

- Proposals:
  - $C(1) = 200$
  - $C(2) = 400$
  - $C(3) = 500$
- Three voters (v_1, v_2 and v_3 respectively):
  - $M_{v_1} = 1000$
  - $M_{v_2} = 800$
  - $M_{v_3} = 600$

Voter 1

- Recall:
  - $C(1) = 200$
  - $C(2) = 400$
  - $C(3) = 500$
  - $M_{v_1} = 1000$
- $C(1) + C(2) + C(3) > M_{v_1}$
  - Voter 1 does not want all three proposals to pass. How do we arrange the 8 outcomes?

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$$P_{v_1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{matrix} 900 \\ 700 \\ 600 \\ 500 \\ 400 \\ 200 \\ 0 \\ 1100 \end{matrix}$$

Voter 2

- Recall:
  - $C(1) = 200$
  - $C(2) = 400$
  - $C(3) = 500$
  - $M_{v_2} = 800$
- $C(1) + C(2) + C(3) > C(2) + C(3) > M_{v_2}$
  - Voter 2 does not want proposals 2 and 3 both to pass, or all three proposals to pass.

Voter 2

- Recall:
  - $C(1) = 200$
  - $C(2) = 400$
  - $C(3) = 500$
  - $M_{v_2} = 800$
- $C(1) + C(2) + C(3) > C(2) + C(3) > M_{v_2}$
  - Voter 2 does not want proposals 2 and 3 both to pass, or all three proposals to pass.

$$P_{v_2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{matrix} 700 \\ 600 \\ 500 \\ 400 \\ 200 \\ 0 \\ 900 \\ 1100 \end{matrix}$$

Voter 3

- Recall:
  - $C(1) = 200$
  - $C(2) = 400$
  - $C(3) = 500$
  - $M_{v_3} = 600$
- $C(1) + C(2) + C(3) > C(2) + C(3) > C(1) + C(3) > M_{v_3}$
  - Voter 3 does not want proposals 1 and 3 both to pass, proposals 2 and 3 both to pass, or all three proposals to pass.

Voter 3

- Recall:
  - $C(1) = 200$
  - $C(2) = 400$
  - $C(3) = 500$
  - $M_{v_3} = 600$
- $C(1) + C(2) + C(3) > C(2) + C(3) > C(1) + C(3) > M_{v_3}$
  - Voter 3 does not want proposals 1 and 3 both to pass, proposals 2 and 3 both to pass, or all three proposals to pass.

$$P_{v_3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{matrix} 600 \\ 500 \\ 400 \\ 200 \\ 0 \\ 700 \\ 900 \\ 1100 \end{matrix}$$

And The Results Are...

$$P_{v_1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} P_{v_2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} P_{v_3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

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- Proposals 1,2 and 3 all pass!

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- Notice that all three proposals passing was the least preferred outcome by every voter.

Notation

- Q denotes the question set $\{1,2,\ldots,n\}$ for $n \in \mathbb{N}$.
- X denotes the set of all outcomes.
  - For $|Q| = 2$, $X = \{00, 10, 01, 11\}$.
- For $x, y \in X$, $x \succ y$ denotes x is preferred to y.
- $C(x)$ denotes the cost of outcome x.
- M_v denotes the maximum amount of money voter v wants to spend.

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Axioms

In order to create a preference matrix for a voter, we must first abide by some simple axioms. The axioms we will adopt are:

Axiom

For $x, y \in X$ such that $C(x), C(y) \leq M_v$, if $C(y) > C(x)$, then $y \succ x$.

Axiom

If $C(x) < C(y)$ and $C(y) > M_v$, then $x \succ y$.

Recall

- How can we model cost-conscious voters?
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Separability

Definition

A set of questions is separable if and only if the ordering on its outcomes is independent of the outcome on the remaining questions.

$$P_v = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

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- If proposal 1 passes:

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  10
  00
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- If proposal 1 passes:

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  10
  00
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- If proposal 1 does not pass:

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  00
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Character

Definition

Let P_v be a preference matrix for an election on Q. Then the **character** of P_v, or $\text{char}(P_v)$ is the collection of all subsets of Q that are separable with respect to P_v.

- From our previous example, $\text{char}(P_v) = \{\emptyset, \{2,3\}, \{1,2,3\}\}$.

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Theorem

For $|Q| = 3$, the only character that is not admissible is $\{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}\}$.

How prevalent are preference matrices that are consistent with our model?

- In all, there are 8!=40,320 possible preference matrices in the 3 question case.
  - Of these 40,320 preference matrices, there are between 2,466 and 3,720 that are consistent with our axioms.
  - Notice that this is between approximately 6% and 9% of all possible preference matrices.

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Theorem

As the number of proposals grows, the percentage of preference matrices that are consistent with our axioms approaches zero.

Condorcet

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Theorem

When no outcomes have the same cost, there is always at least a weak Condorcet winner.

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Theorem

When no outcomes have the same cost, there is always at least a weak Condorcet loser and it is either $11\ldots1$ *or* $00\ldots0$.

Future Work

- What types of interdependencies are present with 4 or more proposals?
- What happens to the model if we relax the assumptions?
- If a Condorcet winner exists, how can we ensure that it is selected?
- How can we model other types of voters found in the general population?

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